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Abstract: In this work, we characterize transfer functions that can be realized with standard electronic components in linearized form, *e.g.* those commonly used in the design of analog amplifiers (including transmission lines) in the small signal regime. We define the stability of such transfer functions in connection with scattering theory, *i.e.* in terms of bounded reflected power against every sufficiently large load. In the simplest model for active elements, we show that unstable transfer functions exist which have no pole in the right half-plane. Then, we introduce more realistic transfer functions for active elements which are passive at very high frequencies, and we show that they have finitely many poles in the right half-plane. Finally, in contrast to the ideal transfer functions studied before, the stability of such "realistic" transfer functions is characterized by the absence of poles in the open right half-plane and the positivity of the real part of the residues of the poles located on the imaginary axis.
This report is written in a way which is suitable to the non-specialist, and every notion is defined and analyzed from first principles.

Key-words: Amplifier, transfer function, active components, transistor, diode, transmission line, negative resistor, stability, Hardy spaces.

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Sur les fonctions de transfert réalisables avec des composants électriques actifs

Résumé : Dans ce travail, nous caractérisons les fonctions de transfert qui peuvent être synthétisées avec des composants électroniques standards linéarisés, y compris des lignes de transmission. Ce sont les composants typiquement utilisés pour la synthèse d'amplificateurs analogiques, modélisés en régime « faible signal ». Nous définissons la stabilité de telles fonctions de transfert en nous appuyant sur la théorie de dispersion des ondes, précisément en demandant à ce que la puissance réfléchie contre toute charge suffisamment grande reste bornée vis-à-vis de la fréquence. Nous montrons qu'il existe des fonctions de transfert qui sont instables mais n'ont pas de pôle dans le demi-plan droit. Nous introduisons ensuite une modélisation plus réaliste des fonctions de transfert des composants actifs, pour traduire l'hypothèse réaliste selon laquelle ils deviennent passifs à très haute fréquence. Nous montrons que les circuits synthétisables avec de tels composants réalistes ont un nombre fini de pôles dans le demi-plan droit ; en outre, nous montrons qu'on peut caractériser la stabilité des fonctions de transfert ainsi obtenues par l'absence de pôle dans le demi-plan droit ouvert et le fait que les résidus des pôles situés sur l'axe imaginaire aient une partie réelle positive.

Ce rapport est écrit de telle façon qu'il soit accessible au non spécialiste et chaque notion est définie et étudiée à partir de notions élémentaires.

Mots-clés : Amplificateur, fonction de transfert, composants actifs, transistor, diode, ligne de transmission, résistance négative, stabilité, espaces de Hardy.

Contents

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1 Introduction

In this work, we characterize transfer functions that can be realized with standard electronic components in linearized form, *e.g.* those commonly used in the design of analog amplifiers (including transmission lines) in the small signal regime. We define the stability of such transfer functions in connection with scattering theory, *i.e.* in terms of bounded reflected power against every sufficiently large load. In the simplest model for active elements, we show that unstable transfer functions exist which have no pole in the right half-plane. Then, we introduce more realistic transfer functions for active elements which are passive at very high frequencies, and we show that they have finitely many poles in the right half-plane. Finally, in contrast to the ideal transfer functions studied before, the stability of such “realistic” transfer functions is characterized by the absence of poles in the open right half-plane and the positivity of the real part of the residues of the poles located on the imaginary axis.

This report is written in a way which is suitable to the non-specialist, and every notion is defined and analyzed from first principles.

2 Electronic components under consideration

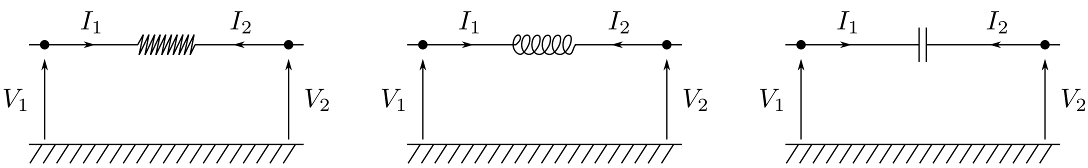
In this section, we give a detailed account of elementary ideal models for electronic components that we consider, along with equations satisfied by currents and voltages at their terminals. These equations are expressed in terms of complex impedances and admittances [2, 3], *i.e.* we express the relations between Laplace transforms of these currents and voltages, see Section 11 for definitions. We denote Laplace transforms with uppercase symbols, *e.g.* $V = V(s)$ is a function of a complex variable s which stands for the Laplace transform of the voltage $v = v(t)$ which is a function of the time t.

By convention, we always orient currents so that they *enter* electronic components.

2.1 Dipoles

Electronic dipoles that we consider in the sequel are the following:

- Ideal resistor, with a positive impedance R.
- Ideal inductor, with impedance of the form Ls ($L > 0$).
- Ideal capacitor, with impedance of the form $1/(Cs)$ ($C > 0$).



(a) Resistor (b) Inductor (c) Capacitor

Figure 1: Symbols for linear dipoles

Relations between the currents I_1 and I_2 entering each terminal of the dipole and the potentials V_1 and V_2 at each terminal are given by

$$\begin{pmatrix} 1 & -1 & 0 & Z \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1}$$

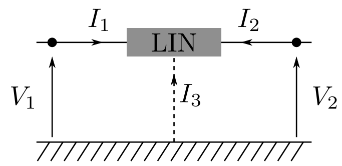
where $Z = \alpha$, $Z = \alpha\, s$ or $Z = \alpha/s$ with $\alpha > 0$.

Remark 1. *It follows directly from Equation* (1) *that* $V_1\,\overline{I_1} + V_2\,\overline{I_2} = Z\,|I_1|^2 = Z\,|I_2|^2$*. Since it is clear that* $\Re Z(s) \geq 0$ *for all components as above, an immediate consequence is that* $\Re(V_1\,\overline{I_1} + V_2\,\overline{I_2}) \geq 0$ *when* $\Re(s) \geq 0$.

2.2 Transmission lines

Transmission lines are distributed components: they are viewed as a series of infinitesimal resistors, capacitors and inductors which is usually modeled by telegrapher's equation [9, sec. 9.7.3], see the discussion in Section 10. All transmission lines in a circuit are assumed to share the same ground. The latter is often implicit and is not drawn along with the symbol for a transmission line. In Section 6, it will be convenient to materialize the current loss between terminals of a line as resulting from a current occurring in a wire (which does not actually exist) connected to the ground. This virtual wire is drawn with a dotted segment on Figure 2. One may ignore it, in which case one should ignore as well the last row and the last column of the matrix in Figure 2.

The behavior of a transmission line is otherwise linear and characterized by the relations in Figure 2. In the matrix shown there, $\gamma = \sqrt{(R + Ls)(G + Cs)}$ is sometimes called the propagation coefficient (note it is frequency-dependent) while $z_0 = (R + Ls)/\gamma$ is the so-called characteristic impedance of the line (*cf.* Section 10). Here R, G, L and C are nonnegative numbers.



$$\begin{pmatrix} -1 & 0 & z_0 \coth(\gamma) & \dfrac{z_0}{\sinh(\gamma)} & 0 \\ 0 & -1 & \dfrac{z_0}{\sinh(\gamma)} & z_0 \coth(\gamma) & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Figure 2: Symbol and relations for transmission lines

2.3 Diodes

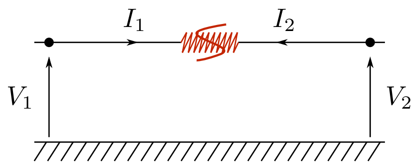
Next we consider diodes. A commonly accepted model of the diode assumes that the relation between the current i traversing the diode and the voltage u is given as a relation $i = f(u)$ where f is a non linear real-valued function. In particular, one assumes that the diode has no inductive nor capacitive effect (f only depends on u and not on $\mathrm{d}u/\mathrm{d}t$ nor $\mathrm{d}i/\mathrm{d}t$). We only study small perturbations around a polarization point $(u^{(Q)}, i^{(Q)})$, hence it is legitimate to linearize the behavior of the diode, which gives us

$$i - i^{(Q)} = g \cdot (u - u^{(Q)}), \quad g = \frac{\mathrm{d}f}{\mathrm{d}u}(u^{(Q)}).$$

Hereafter we rename $i - i^{(Q)}$ as i and $u - u^{(Q)}$ as u. That is, although the variables of interest to the linearized model of the diode are incremental rather than absolute electrical quantities, we denote them like any other intensity or voltage for notational homogeneity. Taking Laplace transforms we get $I = gU$, so the (linearized) diode appears as a standard linear dipole with admittance $g \in \mathbb{R}$. Typical in our context are tunnel diodes which behave (once correctly polarized) as ideal negative resistors: $g < 0$. The symbol we use for, as well as the relations satisfied by linearized diodes are summarized in Figure 3.

2.4 Transistors

Our circuits may also contain transistors. Specifically, we consider field-effect transistors. These have three terminals called gate, source, and drain (denoted respectively by G, S and D). The behavior is usually described by a relation of the form $i_D = f(u_{GS}, u_{DS})$ where f is a non-linear



$$\begin{pmatrix} 1 & -1 & 0 & -\alpha \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

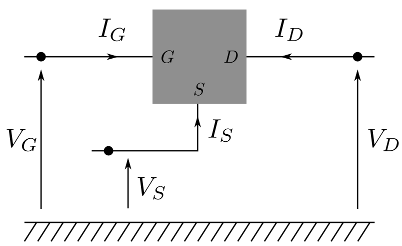
Figure 3: Symbol and relations for the linearized diode ($\alpha > 0$)

real-valued function and $u_{GS} = v_G - v_S$, $u_{DS} = v_D - v_S$ (see Figure 4). As in the case of diodes, this simple model assumes no inductive nor capacitive effect, as f only depends on u_{GS}, u_{DS} and not on their time derivatives, nor on the derivative of i_D. Moreover the function f is increasing in both variables. Another feature of field-effect transistors is that no current enters the gate: $i_G = 0$.

Exactly the same way as for diodes, we consider only small perturbations around a polarization point $(u_{GS}^{(Q)}, u_{DS}^{(Q)}, i_D^{(Q)})$, so we may use a linear approximation:

$$i_D - i_D^{(Q)} = g_m\,(u_{GS} - u_{GS}^{(Q)}) + g_d\,(u_{DS} - u_{DS}^{(Q)}), \quad g_m = \partial_1 f(u_{GS}^{(Q)}, u_{DS}^{(Q)}) > 0 \text{ and } g_d = \partial_2 f(u_{GS}^{(Q)}, u_{DS}^{(Q)}) > 0.$$

Altogether, renaming $i_D - i_D^{(Q)}$ as i_D, $u_{GS} - u_{GS}^{(Q)}$ as u_{GS}, $u_{DS} - u_{DS}^{(Q)}$ as u_{DS} and taking



$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -g_m & -g_d & g_m + g_d & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} V_G \\ V_D \\ V_S \\ I_G \\ I_D \\ I_S \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Figure 4: Symbol and relations for the linearized transistor ($g_m > 0$ and $g_d > 0$)

Laplace transforms as we did for the diode, the (linearized) transistor appears as a current source controlled by a voltage (see Figure 5).

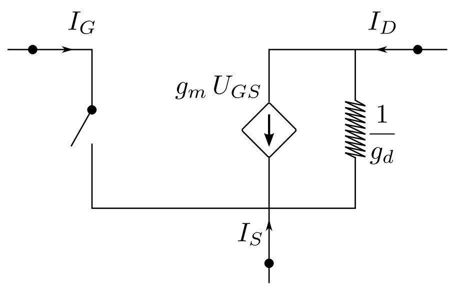


Figure 5: Equivalent circuit for the linearized transistor

3 Structure of circuits

Formally speaking, a circuit is a directed graph with labeled vertices meeting the following constraints:

- There are two kinds of vertices: electronic components and junction nodes.
- A junction node has degree greater or equal to 2.
- An electronic component labeled as a resistor, capacitor, inductor, diode or transmission line has exactly degree 2.
- An electronic component labeled as a transistor has exactly degree 3.
- An electronic component can only be adjacent with a junction node and reciprocally.
- The edges are oriented from junction nodes to electronic components (this definition is non-ambiguous and applies to all edges because of the previous rule).

We number the junction nodes and the edges. Without loss of generality, we can suppose that edges adjacent to a given electronic component are numbered consecutively (because an edge is adjacent to one and only one such component), and that the ordering gate-drain-source prevails in the case of transistors.

To each junction node j is associated a potential V_j and to each edge k is associated an electric current I_k. One junction node is called ground (its potential is 0 by convention and without loss of generality, we suppose that it is numbered as vertex 1). An example of circuit is given in Figure 6: electronic components are represented with their specific symbols introduced in Section 2, but they should now be understood as vertices of the graph. Junction nodes are indicated with bullets, except for the ground which is represented the usual way. For clarity, the ground is represented at multiple places on the figure, but it should be seen as a single vertex.

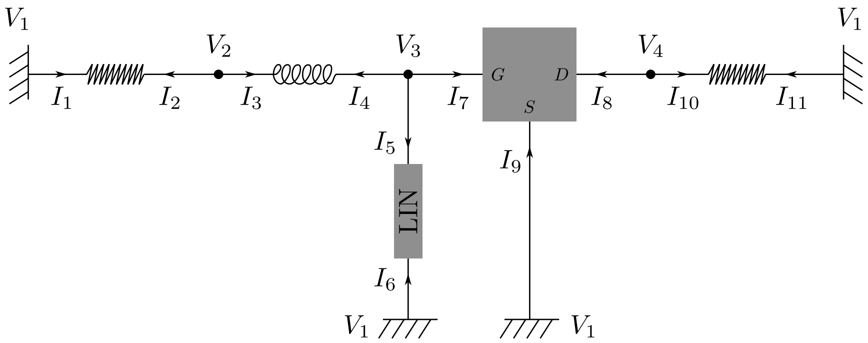


Figure 6: Example of a circuit

Let us denote by $X = {}^t(V_1, \ldots, V_n, I_1, \ldots, I_p)$ the vector made of all potentials and currents of a circuit. Theses quantities are related as follows.

- The ground potential is 0: $V_1 = 0$.
- For each junction node k distinct from the ground, Kirchhoff's first law holds:

$$\sum_{\text{edge } j \text{ adjacent to } k} I_j = 0.$$

The ground is here excluded because currents may exist between transmission lines and the ground, though they are not figured on the graph representing the circuit.

- For each electronic component k, relations from Section 2 between potentials at junction nodes adjacent to the component and currents entering the component must be satisfied.

Collecting all these relations in a matrix, we see that potentials and currents in the circuit must satisfy a relation $MX = 0$, with M a matrix of the form:

$$M = \left(\begin{array}{cc|ccc} 1 & 0 & & 0 & \\ \hline & 0 & & C & \\ \hline & & B_1 & & \\ & A & & \ddots & \\ & & & & B_m \end{array}\right). \tag{2}$$

In equation (2), blocks should be interpreted as follows.

- The first row defines the ground.
- C has $n-1$ rows p columns. Each row expresses an instance of Kirchhoff's law.
- The B_i are 2×2 or 3×3 blocks corresponding to the right-part (*i.e.* multipliers of intensities) of the matrices describing the elementary behavior of each electronic component, as detailed in Section 2.
- A has p rows and n columns. Its elements are those of the left-part (*i.e.* multipliers of voltages) of the matrices describing the elementary behavior of each electronic component.

We call M the behavior matrix of the circuit.

The presence of active components, namely diodes and transistors, may result in M being singular. This well-known fact is illustrated by the example given in Figure 7.

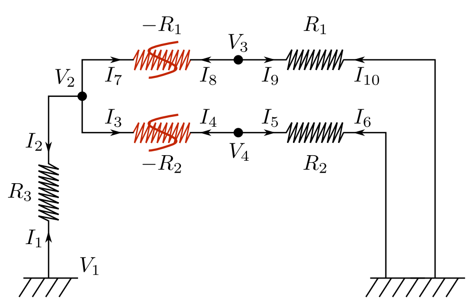


Figure 7: Non-zero currents and potentials may exist in the circuit in the absence of a source

The behavior of this circuit is given by the equation

$$
\left(\begin{array}{cccc|cccccccccc}
1 & 0 & 0 & 0 & \multicolumn{10}{c}{0} \\
\hline
 & & & & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\multicolumn{4}{c|}{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & -1 & 0 & 0 & 0 & R_3 & & & & & & & & \\
0 & 0 & 0 & 0 & 1 & 1 & & & & & & 0 & & \\
0 & 1 & 0 & -1 & & & 0 & -R_2 & & & & & & \\
0 & 0 & 0 & 0 & & & 1 & 1 & & & & & & \\
-1 & 0 & 0 & 1 & & & & & 0 & R_2 & & & & \\
0 & 0 & 0 & 0 & & & & & 1 & 1 & & & & \\
0 & 1 & -1 & 0 & & & & & & & 0 & -R_1 & & \\
0 & 0 & 0 & 0 & & & & & & & 1 & 1 & & \\
-1 & 0 & 1 & 0 & & & 0 & & & & & & 0 & R_1 \\
0 & 0 & 0 & 0 & & & & & & & & & 1 & 1
\end{array}\right)
\left(\begin{array}{c}
V_1 \\ V_2 \\ V_3 \\ V_4 \\ \hline I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \\ I_8 \\ I_9 \\ I_{10}
\end{array}\right) = 0.
$$

One easily sees that, for any I, the vector ${}^t(0,\, 0,\, R_1 I,\, -R_2 I,\, 0,\, 0,\, -I,\, I,\, -I,\, I,\, I,\, -I,\, I,\, -I)$ is a solution, which proves that the matrix M in this example is not invertible.

Now, a non-invertible behavior matrix corresponds to a situation where, in the absence of current or voltage source in the circuit, a non-trivial equilibrium between currents and voltages can be established. In other words, energy transfer occurs between active and passive parts of the circuit, without external excitation. Such a property is clearly undesirable, for it entails that the response of the circuit to external excitations is not uniquely determined by these but also depends on certain unobservable endogenous quantities. From the point of view of design, it indicates that the latter should be revisited in order to suppress useless loss of energy. We will suppose in the rest of this work that the behavior matrix M in Equation (2) is invertible.

4 Partial transfer functions

The (local) stability of a circuit is studied by observing how it responds to small perturbations. The latter can be either a set of small current sources at junction nodes or a set of small voltage sources at terminals of the components. Due to smallness of the hypothesized perturbations, one considers the linearized model whose behavior, when viewed as a system whose inputs are the perturbations and whose outputs are a set voltages or currents in the circuit, will determine whether the latter is (locally) stable or not. We are thus led to build the transfer function of this system (see Section 11), which is a matrix whose entries are elementary transfer functions corresponding to a single perturbation (input) applied at some junction node (in case of a current source perturbation) or edge (in case of a voltage source perturbation) and whose effect (output) is observed at some node or edge. These we call partial transfer functions (or partial frequency responses) of the circuit, reserving the name transfer function (or transfer matrix for emphasis) for the full matrix of all partial transfer functions of the circuit. In this section we review how to compute partial transfer functions, and in later sections we shall characterize them.

4.1 Partial transfer from a current source

Some ideal current source I is plugged in between the ground and some junction node α.

Plugging the current source changes Kirchhoff's law at α: it becomes $\sum_j I_j = I$, where the sum is taken over all edges j adjacent to α. The behavior of the perturbed circuit is thus given by

$$MX = \begin{pmatrix} 0 \\ \vdots \\ I \\ \vdots \\ 0 \end{pmatrix}.$$

If M is invertible we can therefore write

$$\begin{pmatrix} V_1 \\ \vdots \\ V_n \\ I_1 \\ \vdots \\ I_p \end{pmatrix} = M^{-1} \begin{pmatrix} 0 \\ \vdots \\ I \\ \vdots \\ 0 \end{pmatrix},.$$

It is hence clear that, for any k, $V_k = M^{-1}_{[k,\beta]}\, I$, where β is the index of that row in M expressing Kirchhoff's law at node α. Here, $M^{-1}_{[i,j]}$ denotes the entry at row i and column j of M^{-1}. We see that the voltage at any node k depends linearly on I. The ratio between (the Laplace transform of) the voltage V_k and I is the partial transfer function (or frequency response) of the circuit from current at α to voltage at node k. By Cramer's rule

$$V_k = (-1)^{k+\beta}\, \frac{M_{\beta,k}}{\det M}\, I,$$

where $M_{i,j}$ denotes the minor of M obtained by deleting row i and column j.

Note that invertibility of the behavior matrix is necessary and sufficient for all partial transfer functions from a current source to exist.

4.2 Partial transfer from a voltage source

An edge α is chosen, which goes from a junction node k to some component. An ideal voltage source with Laplace transform U is plugged between k and that component. This changes the potential at the corresponding terminal of the component, which becomes $V_k + U$. The only coefficients in M that require change are those in the k-th column corresponding to a row describing the behavior of the component involved. Examination of the left-part of the matrices described in Section 2 shows that there is only one row which is affected, and that the corresponding coefficient in M is a nonzero real number γ (equal to 1 or -1 in the case of a resistor, an inductor, a capacitor, a diode or a transmission line, and to $-g_m$, $-g_d$ or $g_m + g_d$ in the case of transistor). Therefore we can write for the perturbed circuit

$$MX = \begin{pmatrix} 0 \\ \vdots \\ -\gamma\, U \\ \vdots \\ 0 \end{pmatrix}.$$

Denoting with β the index of the row where $-\gamma U$ lies in the above equation, we get using the same argument as before that, if matrix M is invertible, then

$$I_r = (-1)^{r+\beta} \frac{M_{\beta,r}}{\det M} (-\gamma)U, \quad \text{for any } r \in \{1, \dots, p\}.$$

The ratio I_r/U is the frequency response at edge r of the circuit to the voltage source U, and again invertibility of the behavior matrix is necessary and sufficient for all partial transfer functions from a voltage source to exist.

Altogether, we see that the partial transfer function obtained by using a voltage source is of the same form as the partial transfer function obtained using a current source. In the rest of this work, without loss of generality, we only deal with the latter, that is, we favor transfer functions of impedance type.

5 R-L-C circuits with negative resistors

As a first step towards our main results, we describe in this section the structure of partial transfer functions of circuits that have no transmission line nor transistor in their components. Namely, *each partial frequency response of a circuit made of positive resistors, negative resistors, along with standard (i.e. positive) capacitors and inductors, belongs to the field* $\mathbb{R}(s)$ *of rational functions in the variable* s *with real coefficients, and conversely any* $f \in \mathbb{R}(s)$ *can be realized as a partial frequency response of such a circuit.* Moreover, we will see (*cf.* Remark 2 at the end of the present section) that the result still holds if transistors are added to the list of admissible components.

The above statement is essentially equivalent to a classical property of impedance (rather than partial transfer functions) of networks comprising positive resistors, negative resistors, capacitors and inductors [4]. The latter is formally stated as Theorem 1 in Section 5.2 to come.

Note that the converse part of the statement is concerned with a single partial transfer function, and says nothing about synthesizing an arbitrary rational matrix as the transfer matrix of a circuit made of resistors of arbitrary sign, capacitors and inductors. Whether this is possible or not is still an open issue, see [4] where transformers and gyrators are added to the set of admissible elements in order to answer the question in the positive.

We now discuss the proof. The fact that each partial frequency response belongs to $\mathbb{R}(s)$ is obvious from the previous section, for this frequency response has the form $\alpha\, M_{\beta,k}/\det(M)$ with $\alpha \in \mathbb{R}$, and M as in equation (2). Besides, since the circuit does not contain transmission lines, all elements of M belong to $\mathbb{R}(s)$, thus also $M_{\beta,k}/\det(M) \in \mathbb{R}(s)$.

The converse part is a little harder, and can be established along the classical lines of Foster synthesis [3, thm. 5.2.1] by relaxing sign conditions therein, see also [4]. Below, we give a different proof which lends itself better to generalization when we consider circuits with transmission lines, as will be the case in a forthcoming section.

We make extensive use of the widget given in Figure 8, called an inverter. It is made of two dipoles with impedance X, one with impedance $-X$ and one with impedance Z. Using parallel and series composition rules, one easily sees that it is equivalent to a dipole whose impedance is

$$X + \frac{-X\,(X+Z)}{-X+(X+Z)} = \frac{-X^2}{Z}.$$

Of course, the inverter cannot be realized from passive devices because it is impossible to realize both a network with impedance X and a network with impedance $-X$ with passive components. Having negative resistors at our disposal is thus crucial at this point. Hereafter, we simply speak of impedance of a network to mean impedance of a two-terminal network.

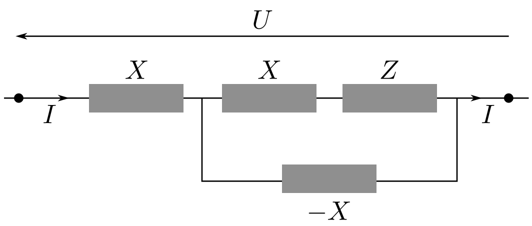


Figure 8: The inverter network

5.1 What the inverter makes possible

Lemma 1. *If negative resistors are allowed, and if one has a network with impedance x, it is possible to build a network with impedance $-1/x$.*

Proof. The inverter of Figure 8 can actually be realized with $X = 1$ and $-X = 1$ since we have negative resistors. Using the network with impedance x for Z, we obtain a network with equivalent impedance $-1/Z = -1/x$. □

Corollary 1. *Having positive and negative resistors, along with standard capacitors and inductors allows one to emulate negative capacitors and inductors.*

Proof. Applying Lemma 1 to a capacitor of capacitance α (i.e. $x = 1/(\alpha\, s)$) gives us a negative inductor of inductance $-\alpha$. Conversely, applying the lemma to an inductor of inductance α (i.e. $x = \alpha\, s$) gives us a negative capacitor of capacitance $-\alpha$. □

Lemma 2. *If negative resistors are allowed and if one has an electrical network with impedance x and another one with impedance $-x$, it is possible to build a network with impedance $\pm x^2$.*

Proof. We use the inverter again, this time with $X = x$. This is possible because we can actually build a network with impedance $-X$. For Z, we can choose as a resistor of resistance ± 1 (since negative resistors are allowed). We hence obtain a network with equivalent impedance $-X^2/\pm 1 = \pm x^2$. □

Lemma 3. *If negative resistors are allowed and if one has electrical networks with impedances x, $-x$, y and $-y$, it is possible to build a network with impedance $\pm xy$.*

Proof. Composing networks for x and y (respectively $-x$ and $-y$) in series, we get a network of impedance $x + y$ (respectively $-(x + y)$). Using Lemma 2, we get circuits with impedances $\pm(x + y)^2$, $\pm x^2$ and $\pm y^2$.

Now, composing in series $(x + y)^2$, $-x^2$ and $-y^2$, we get a network of equivalent impedance $(x + y)^2 + (-x^2) + (-y^2) = 2xy$. The same way, we also get a circuit with impedance $x^2 + y^2 + (-(x + y)^2) = -2xy$.

To sum up, we just proved that, having networks with impedances $\pm x$ and $\pm y$, one can build networks with impedance $\pm 2xy$. Applying this result to $\pm 2xy$ and a resistor of resistance $\pm 1/4$, we get a network with impedance $\pm xy$. □

5.2 Frequency response of RLC circuits with negative resistors

Theorem 1. *Let $P(s)/Q(s)$ be an arbitrary rational function with real coefficients. Using positive and negative resistors, capacitors and inductors, it is possible to build a network with impedance $P(s)/Q(s)$.*

Proof. Using Corollary 1 together with Lemma 3, we see by an elementary induction that we can build networks with impedances $\pm\alpha s^k$ for any $k \in \mathbb{Z}$. Composing them in series allows us to realize $\pm P(s)$ and $\pm Q(s)$. Now, using Lemma 1, we can realize $\pm 1/Q(s)$, and finally, using Lemma 3 again, we get $\pm P(s)/Q(s)$. $\square$

Let finally $P(s)$ and $Q(s)$ be two polynomials such that $0 \not\equiv Q \neq P$. To establish the result announced at the beginning of this section, it remains to observe in view of Theorem 1 that the circuit shown in Figure 9, where we set $R(s) = P(s)/Q(s)$, is realizable with positive and negative resistors, inductors, capacitors. Indeed, it consists of a series of two elements with impedance 1 and $R(s)$ respectively, with both ports connected to the mass. Now, one easily checks that the partial frequency response to a current source plugged at the junction node with output the voltage at that node is exactly $P(s)/Q(s)$. This achieves the proof.

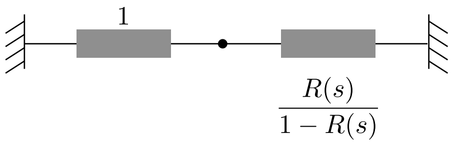


Figure 9: A circuit with partial frequency response $R(s)$

Remark 2. *Circuits made of positive and negative resistors, capacitors, inductors and linearized transistors have exactly the same class of partial frequency responses as those without transistors. Indeed, elements in the matrix describing the behavior of transistors (see Figure 4) also belong to $\mathbb{R}(s)$, so the frequency response of circuits with transistors in turn lies in $\mathbb{R}(s)$. Since all functions from $\mathbb{R}(s)$ are already realizable without transistors, this remains* a fortiori *true when transistors are allowed.*

6 Circuits with transmission lines

Below, we generalize the result of the previous section to the case where circuits consist of all elements listed in Section 2. More precisely, let $\mathcal{E}$ be the smallest field containing $\mathbb{R}(s)$ as well as all functions of the form $\gamma(s)\sinh(\gamma(s))$ and $\cosh(\gamma(s))$, where $\gamma(s) = \sqrt{(a+bs)(c+ds)}$ for some real numbers $a, b, c, d \geq 0$. The determination of the square root involved in the expression for γ is irrelevant: choosing a determination or its negative defines the same functions because cosh is even and sinh is odd. Another way of defining $\mathcal{E}$ is, e.g.,

$$\mathcal{E} = \{(f_1 + \cdots + f_n)/(g_1 + \cdots + g_m),\ \text{with}\ f_1, \ldots, f_n, g_1, \ldots, g_m \in \mathcal{A}\} \qquad \text{where}$$

$$\mathcal{A} = \left\{ \alpha\, s^k \prod_{i=1}^{n} \gamma_i(s)\, \sinh(\gamma_i(s)) \prod_{i=n+1}^{m} \cosh(\gamma_i(s)) \right\}_{\substack{\alpha\in\mathbb{R},\, k\in\mathbb{N}, \\ \gamma_i(s)=\sqrt{(a_i+b_i s)(c_i+d_i s)} \\ \text{where } a_i, b_i, c_i, d_i \geq 0}}. \tag{3}$$

The main result of this section is that *the class of functions realizable as partial transfer functions of circuits made of elements listed in Section 2, namely positive and negative resistors, capacitors, inductors, linearized transistors and transmission lines, is exactly $\mathcal{E}$.*

We follow the same approach as in the previous section. Again, proving that each partial frequency response belongs to $\mathcal{E}$ is very easy: all entries of the matrix M given in Equation (2) belong to $\mathcal{E}$, which is a field, thus any partial frequency response (which is proportional to the ratio of a minor of M and its determinant) also belongs to $\mathcal{E}$.

We now show the converse. For this, we use a trick: we only consider networks with one available terminal, for which a relation of the form $V = Z\,I$ is satisfied, where V is the potential of the terminal and I is the current entering the network through the terminal. We call such a network a one-port circuit and we call Z the impedance of the circuit.

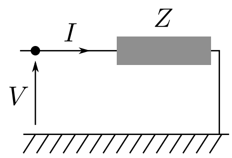


Figure 10: Representation of a one-port circuit

Of course such circuits form a strict, and in fact very particular subset of all circuits one can built from positive or negative resistors, inductors, capacitors and transmission lines. One could think *a priori* that partial frequency responses realizable in this way are only a small subset of all partial frequency responses arising from more general topologies. As we will see this is not the case, as all functions of $\mathcal{E}$ can be realized as impedances of one-port circuits already. Now, if $R(s)$ is an element of $\mathcal{E}$, so is $R(s)/(1 - R(s))$, and we can use again the the circuit of Figure 9. Indeed, it can be synthesized with a one-port circuit of impedance $R(s)/(1 - R(s))$ and a resistor of resistance 1. One easily checks that the partial frequency response obtained by plugging a current source at the junction node and looking at the voltage at that same node is again $R(s)$.

6.1 Using transmission lines as one-port circuits

The reason why we limit our study to one-port circuits is the following. Whereas dipoles are easy to compose in series or parallel and this composition results in nice algebraic combinations of their impedance functions, it is not so for transmission lines. Their behavior, recalled in Figure 2, cannot be reduced to a single scalar relation, and one fundamentally needs two linear relations to express it. Thus, when composing lines with other elements in a circuit, one is led to multiply 2×2 matrices from which it is not easy to keep track of the algebraic structure of the resulting elements.

In contrast, using a transmission line as a one-port circuit by forcing either the current or the potential at one of its terminals simplifies the relations and allows us to retain a single linear equation to represent its behavior in the form of the impedance of a one-port circuit. This we see from the following two lemmas.

Lemma 4. *Let a, b, c, and d be four nonnegative numbers. Let us set $\gamma(s) = \sqrt{(a + bs)(c + ds)}$, where the determination of the square root is arbitrary. It is possible to realize a one-port circuit with impedance $(a + bs)/\gamma(s) \cdot \coth(\gamma(s))$.*

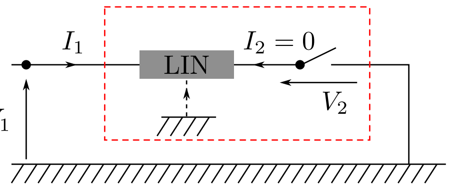


Figure 11: One-port circuit with impedance $(a + bs)/\gamma(s) \cdot \coth(\gamma(s))$

Proof. Let us consider a transmission line with characteristics $R = a$, $L = b$, $G = c$, and $C = d$. If the second terminal of the line is left open, I_2 is forced to 0. Using the relations given in Figure 2, we get

$$V_1 = \frac{a+bs}{\gamma(s)} \coth(\gamma(s))\, I_1 + \frac{a+bs}{\gamma(s)\,\sinh(\gamma(s))}\, I_2 = \frac{a+bs}{\gamma(s)} \coth(\gamma(s))\, I_1$$

□

Lemma 5. *Let a, b, c, and d be four nonnegative numbers. Let us set $\gamma(s)$ as in previous lemma. It is possible to realize a one-port circuit with impedance $\gamma(s)/(a+bs)\cdot\tanh(\gamma(s))$.*

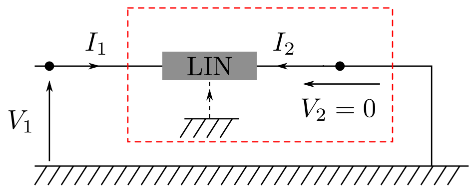


Figure 12: One-port circuit with impedance $\gamma(s)/(a+bs)\cdot\tanh(\gamma(s))$.

Proof. Let us consider a transmission line with characteristics $R = c$, $L = d$, $G = a$, and $C = b$. Remark that $\gamma(s)$ is still equal to $\sqrt{(a+bs)(c+ds)}$. This time, we connect the second terminal directly to the ground. This forces $V_2 = 0$. Using the second relation given in Figure 2, we get

$$\frac{c+ds}{\gamma(s)\,\sinh(\gamma(s))}\, I_1 + \frac{c+ds}{\gamma(s)} \coth(\gamma(s))\, I_2 = V_2 = 0,$$

and hence $I_2 = -I_1/\cosh(\gamma(s))$.

The other relation is

$$V_1 = \frac{c+ds}{\gamma(s)} \coth(\gamma(s))\, I_1 + \frac{c+ds}{\gamma(s)\,\sinh(\gamma(s))}\, I_2 = \frac{c+ds}{\gamma(s)} \left(\frac{\cosh(\gamma(s))}{\sinh(\gamma(s))} - \frac{1}{\sinh(\gamma(s))\,\cosh(\gamma(s))} \right) I_1.$$

Since $\cosh(\gamma(s))^2 - 1 = \sinh(\gamma(s))^2$, the expression simplifies to $V_1 = (c+ds)/\gamma(s)\cdot\tanh(\gamma(s))\, I_1$. We conclude by remarking that $(c+ds)/\gamma(s) = \gamma(s)/(a+bs)$. □

Remark 3. *Let us consider a two-ports network with impedance $R(s)$. Connecting one of the terminals of the network to the ground, we get a one-port circuit. The potential V at the remaining terminal is the voltage between both terminals of the network, since the ground has potential 0. By definition, $V = R(s)\, I$, hence the one-port circuit so obtained also has impedance $R(s)$.*

6.2 Composing one-port circuits

Composing one-port circuits with other electrical elements is a little more complicated than composing dipoles, since only one terminal remains available. Still, we can compose one-ports circuits in parallel or compose a one port-circuit in series with a dipole: this indeed works as expected which is shown in the following two lemmas.

Lemma 6 (Parallel composition of one-port circuits)**.** *If one has a one-port circuit with impedance x and a one-port circuit with impedance y, it is possible to build a one-port circuit, with impedance $xy/(x+y)$.*

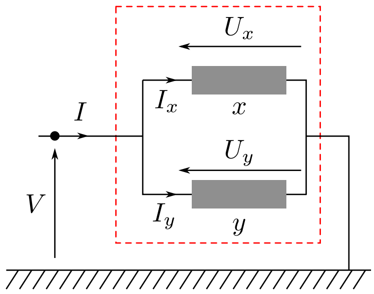


Figure 13: Composition of one-port circuits in parallel

Proof. We simply connect both circuits in parallel. Then $I = I_x + I_y$ by Kirchhoff's law and $V = U_x = U_y$. Moreover, by hypothesis $U_x = x\, I_x$ and $U_y = y\, I_y$. We deduce that

$$I = \frac{U_x}{x} + \frac{U_y}{y} = \left(\frac{1}{x} + \frac{1}{y}\right) V = \frac{y+x}{xy}\, V,$$

hence the result. □

Lemma 7 (Series composition of a one-port circuit and a dipole)**.** *If one has a one-port circuit with impedance x, it is possible to build a one-port circuit with impedance $x + R(s)$, for any rational function R with real coefficients.*

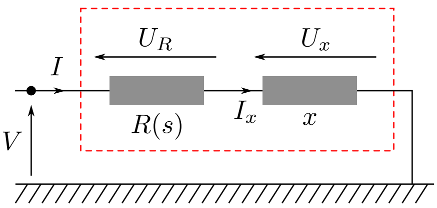


Figure 14: Composition of a one-port circuit and a dipole in series

Proof. Let us consider a dipole of impedance $R(s)$, as given by Theorem 1 and let us connect it in series with the one-port circuit of impedance x.Then $I = I_x$ because the current is preserved from one terminal of the dipole to the other. Moreover $U_R = R(s)\, I$ and $U_x = x\, I_x$ by hypothesis. Therefore, $V = U_x + U_R = (x + R(s))\, I$. □

There is obviously a problem to compose one-port circuits in series: by construction, one-port circuits have only one terminal available, which prevents us from chaining them to add their impedances. Yet, we remark that Lemma 1 still holds if "network" is now understood as "one-port circuit". Indeed, according to Lemmas 6 and 7, the inverter of Figure 8 is realizable with resistors $X = 1$ and $-X = -1$ and a one-port circuit of impedance $Z = x$. This is sufficient to emulate composition in series:

Lemma 8. *If one has a one-port circuit with impedance x and another one with impedance y, it is possible to build a one-port circuit with impedance $x + y$.*

Proof. Using Lemma 1 it is possible to build one-port circuits with impedances $-1/x$ and $-1/y$. Hence, using Lemma 6 we get a circuit with impedance

$$\frac{(-1/x)(-1/y)}{(-1/x)+(-1/y)} = \frac{-1}{y+x}.$$

Using Lemma 1 again, we get a one-port circuit with impedance $x+y$. □

Now that composition in series has been proved to be possible, we see that Lemmas 2 and 3 hold for one-port circuits as well, and we shall be able to proceed much in the same way as we did to establish Theorem 1.

6.3 Class of all impedances of one-port circuits

Lemma 9. *Let a, b, c, and d be four nonnegative numbers. Let us set $\gamma(s) = \sqrt{(a+bs)(c+ds)}$, where the determination of the square root is arbitrary. It is possible to realize one-port circuits with impedances $-(a+bs)/\gamma(s)\cdot\coth(\gamma(s))$ and $-\gamma(s)/(a+bs)\cdot\tanh(\gamma(s))$.*

Proof. According to Lemma 5 (respectively, Lemma 4) there is a one-port circuit with impedance $\gamma(s)/(a+bs)\cdot\tanh(\gamma(s))$ (respectively $(a+bs)/\gamma(s)\cdot\coth(\gamma(s))$). Using Lemma 1 we get a one-port circuit of impedance $-(a+bs)/\gamma(s)\cdot\coth(\gamma(s))$ (respectively $-\gamma(s)/(a+bs)\cdot\tanh(\gamma(s))$). □

Lemma 10. *Let a, b, c, and d be four nonnegative numbers. Let us set $\gamma(s)$ as in previous lemma. It is possible to realize one-port circuits with impedances $\pm\cosh(\gamma(s))^2$.*

Proof. According to Lemmas 5 and 9, there are one-port circuits with impedances $\pm\gamma(s)/(a+bs)\cdot\tanh(\gamma(s))$. Using Lemma 2, we can hence build one-port circuits with impedances

$$\pm\frac{\gamma(s)^2}{(a+bs)^2}\tanh(\gamma(s))^2 = \pm\frac{c+ds}{a+bs}\tanh(\gamma(s))^2.$$

Now, by Remark 3 and Theorem 1, there are one-port circuits with impedances $\pm(a+bs)/(c+ds)$. Therefore, using Lemma 3, we can build one-port circuits with impedances $\pm\tanh(\gamma(s))^2$. Using Lemma 7 we get impedances $\pm(\tanh(\gamma(s))^2-1)$, and finally, using Lemma 1, we get impedances

$$\pm\frac{1}{1-\tanh(\gamma(s))^2} = \pm\cosh(\gamma(s))^2.$$

□

Theorem 2. *Any element of the set $\mathcal{E}$ defined in Equation (3) can be realized as the impedance of a one-port circuit.*

Proof. It is enough to prove that, for any nonnegative numbers a, b, c, and d, one can realize $\pm\gamma(s)\sinh(\gamma(s)$ and $\pm\cosh(\gamma(s))$, where $\gamma(s)$ is defined as in Lemma 9. Indeed, this is sufficient to show that any element of $\mathcal{A}$ is realizable (using Remark 3 and Lemma 3). Then, Lemma 8 allows us to realize any sum of elements of $\mathcal{A}$, and we conclude using Lemmas 1 and 3.

Let us consider arbitrary nonnegative numbers a, b, c, and d. We define $\gamma(s)$ as before: $\gamma(s) = \sqrt{(a+bs)(c+ds)}$. We set $a' = a/2$, $b' = b/2$, $c' = c/2$ and $d' = d/2$ and $\gamma'(s) = \sqrt{(a'+b's)(c'+d's)}$. We obviously have $\gamma(s) = \pm 2\gamma'(s)$ (the determinations of the square roots in the definitions of γ and γ' are not necessarily the same). According to Lemma 10, there are one-port circuits with impedances $\pm\cosh(\gamma'(s))^2$. Now, using the identity $\cosh(2x) = 2\cosh(x)^2-1$, we see that $\cosh(\gamma'(s))^2 = \frac{1}{2}\cosh(\pm\gamma(s)) + \frac{1}{2}$. Using Lemma 7, and since cosh is an even function, we get impedances $\pm\frac{1}{2}\cosh(\gamma(s))$, and by addition we can get rid of the factor 1/2.

Furthermore, since we have impedances $\pm\frac{1}{2}\cosh(\gamma(s))$ and $\pm\frac{\gamma(s)}{a+bs}\tanh(\gamma(s))$, we get

$$\pm\frac{\gamma(s)}{2(a+bs)}\sinh(\gamma(s))$$

by Lemma 3. Using Remark 3 and Theorem 1, there are one-port circuits with impedances $\pm 2(a+bs)$. Thus, by Lemma 3 again, we can build one-port circuit with impedance $\pm\gamma(s)\sinh\gamma(s)$, as desired. □

Remark 4. *Note that each element of $\mathcal{E}$ is a meromorphic function on $\mathbb{C}$, for branchpoints like $-a_i/b_i$, and $-c_i/d_i$ in Equation* (3) *(*cf. *page 13), which are a priori of order 2, are in fact artificial by evenness of $s \mapsto \gamma(s)\sinh\gamma(s)$ and $s \mapsto \cosh\gamma(s)$.*

7 Notion of stability

It is natural to say that a circuit is (locally) stable with respect to small current perturbations if every partial transfer function is stable, and similarly for voltages. In other words, stability should refer to the collection of all partial transfer functions (*i.e.* to the transfer matrix). In practice, however, the computational complexity involved with plugging current sources at every junction node (or voltage sources at every edge) and checking their effect on each current or voltage in the circuit often lies beyond computational and experimental capabilities. Typically, one is content with checking stability on a couple of well-chosen partial transfer functions. We do not lean on the issue of how to pick those, but we shall discuss stability of a single partial transfer function from a current source at a node to a voltage at a node.

The standard definition of stability for a linear dynamical system is that it maps input signals of finite energy (*i.e.* of bounded L^2-norm, both in time and frequency domain since the Fourier transform is an isometry) to output signals of finite energy. This type of stability is denoted as BIBO, that stands for *Bounded Input Bounded Output.* Equivalently, a linear dynamical system is stable if its transfer function belongs to the Hardy space $\mathcal{H}^\infty$, *i.e.* if it is holomorphic and bounded in the right half-plane (see Section 11). This definition is not satisfactory here, for it would term unstable such simple passive components as pure inductors (*i.e.* $Z = Ls$), pure capacitors (*i.e.* $Z = 1/(Cs)$), or ideal transmission lines (those for which when $R = G = 0$ in Section 2.2). To circumvent this difficulty, we first argue that no current source is ever ideal: it always has internal resistance, that in the near to ideal case can be considered as a very large resistor connecting the ground to the node where we plug the current source. Next, we take a hint from scattering theory: if Z is the partial transfer function and $R > 0$ is the load of the current source, then $(R - Z)/(R + Z)$ is the transfer-function from the incoming (maximum available) power wave to the reflected power wave. In other words, $|(R-Z(i\omega))/(R+Z(i\omega))|^2$ is that fraction of the maximum power that the current source can supply to the system which bounces back, at frequency ω, and if I is the intensity of the current then $(1 - |(R - Z(i\omega))/(R + Z(i\omega)|^2)RI^2/4$ is the power actually dissipated by Z (see [8]).

When the circuit is passive, then $\Re Z \geq 0$ hence the fraction of reflected power is less than 1, that is to say $(R - Z)/(R + Z)$ lies in $\mathcal{H}^\infty$ and its supremum norm is at most 1. This indicates that the system does not generate energy. When the circuit is active, which is the case when it contains diodes or transistors, the reflected fraction of the incoming power can be greater than 1 at some or all frequencies, which means that the system generates energy at those frequencies. For instance, an amplifier is expected to magnify the signal it receives. Of course, the necessary power supply to do this has to come from an external source, used to generate voltage at terminals of the primary circuits of diodes and transistors. Thus, even if it has norm greater than unity, $(R-Z)/(R+Z)$ should still lie in $\mathcal{H}^\infty$ to prevent instabilities, namely, working rates for which the energy demand to these primary circuits becomes infinite. In view of this, it seems natural to say that Z is stable if $(R-Z)/(R+Z) \in \mathcal{H}^\infty$. However, we do not want the degree of stability of Z to depend on the actual value of the load, which leads us to the make the following definition which we could not locate in the literature.

Definition 1. *Let Z be a partial frequency response of a circuit. We say that Z is stable if there exists $R_0 > 0$ and $M > 0$ such that*

$$\forall R > R_0, \qquad \frac{R-Z}{R+Z} \in \mathcal{H}^\infty \quad \text{and} \quad \left\| \frac{R-Z}{R+Z} \right\|_{\mathcal{H}^\infty} \leq M. \tag{4}$$

The same definition applies to the impedance of a two-ports.

In other words, our definition states that a circuit is stable, if in terms of power waves it is of BIBO type, and this for all near to perfect feeding current sources (i.e $R \geq R_0$). Definition 1 of stability is more general than the BIBO condition on Z:

Lemma 11. *If $Z \in \mathcal{H}^\infty$ (i.e. if Z is stable in the usual sense), then Z is also stable in the sense of Definition 1.*

Proof. Since $Z \in \mathcal{H}^\infty$ it holds that $|Z(s)| < M$ for some M and all s with $\Re s > 0$. If we set $R_0 = M+1$, then $|Z(s)+R| \geq 1$ for all $R \geq R_0$, so that $|(R-Z(s))/(R+Z(s))|$ is bounded above by $R+M$, implying that it lies in $\mathcal{H}^\infty$. □

Note that the converse of Lemma 11 is not true, *e.g.* ideal inductors and capacitors becomes stable with Definition (4), although they are not themselves in $\mathcal{H}^\infty$.

As pointed out in Section 5, circuits made of positive resistors, negative resistors, capacitors and inductors have rational partial transfer functions, in which case Definition 1 simply says that the rational function $(R-Z)/(R+Z)$ has no pole in the closed right half-plane for all R large enough, including at infinity. In fact, this function cannot have poles at infinity anyway (*i.e.* it is proper), because either Z has a pole there and then $(R-Z)/(R+Z)(\infty) = -1$, or else $Z(\infty) = a \in \mathbb{C}$ in which case $(R-Z)/(R+Z)$ has finite value at infinity for each $R > |a|$. Thus, for such circuits, stability simply means that $(R-Z)/(R+Z)$ has no no pole at finite distance in the closed right half-plane, whenever R is large enough. This familiar criterion for stability no longer holds when lines are present in the circuit, as follows from the example below. Set

$$f(s) = s \tanh(s) - \frac{1}{s+1} \quad \text{and} \quad Z(s) = \frac{2f(s)}{f(s)+2}. \tag{5}$$

We will show that for all R large enough, $s \mapsto (R-Z(s))/(R+Z(s))$ is defined and finite on $\{\Re s \geq 0\}$ but does not belong to $\mathcal{H}^\infty$. This will provide us with an example of a transfer function which is unstable in the sense of Definition 1, yet has no poles in the right half-plane. We proceed *via* the following steps.

- First, the function Z is realizable since it obviously belongs to the set $\mathcal{E}$ defined in Equation (3). In fact, one can check that the circuit shown in Figure 15 has partial transfer function Z from a current source between the ground and the black bullet (bottom-left of the circuit) to the voltage at that bullet.

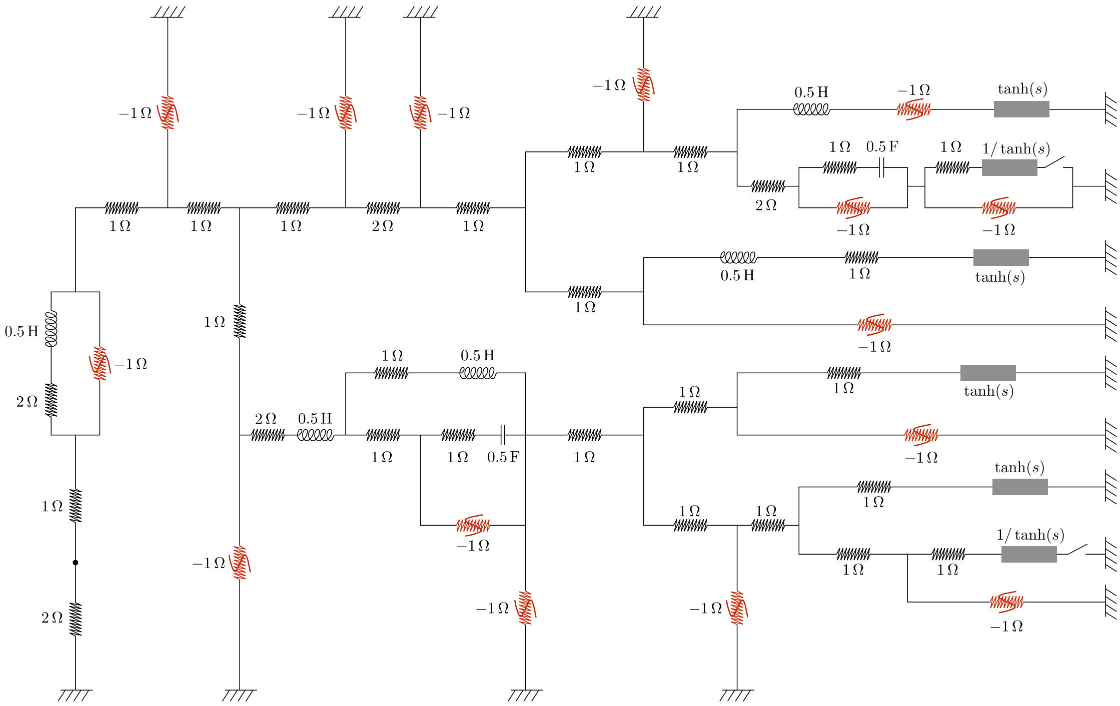


Figure 15: Realization of a function that is unstable, though it has no pole in the right half-plane.

- Next, we remark that the equation $f(s) = -a$ where $a \in [1, +\infty[$ has no solution s in the right half-plane. Indeed,

$$f(s) = -a \iff \tanh(s) + \frac{1}{s+1} = \frac{1-a}{s}.$$

  We observe that $\tanh(s) + 1/(s+1)$ has a positive real part whenever $\Re(s) \geq 0$ though $(1-a)/s$ has a non-positive real part. Thus they cannot be equal.

- Consequently, for all $R > 2$, the function $(R-Z)/(R+Z)$ is finite at every point of the closed right half-plane. Indeed, since

$$\frac{R-Z}{R+Z} = \frac{R-2}{R+2} \cdot \frac{f(s) + \frac{2R}{R-2}}{f(s) + \frac{2R}{R+2}}, \tag{6}$$

  it follows from the previous item that the denominator cannot vanish for $\Re s \geq 0$ and, though f may have poles there, the function $(R-Z)/(R+Z)$ is analytically continued at them with value $(R-2)/(R+2)$.

- To recap, we just showed that for any $R > 2$ the function $(R-Z)/(R+Z)$ is holomorphic in $\{\Re s \geq 0\}$. Still, we claim that it does not belong to $\mathcal{H}^\infty$. Indeed, put for simplicity $\alpha = 2R/(R+2)$ and consider the sequence of points $s_k = \mathrm{i}\,(k\pi + \alpha/(k\pi))$. Then

$$s_k \tanh(s_k) = -\left(k\pi + \frac{\alpha}{k\pi}\right) \tan\left(\frac{\alpha}{k\pi}\right) = -\alpha + \circ\left(\frac{1}{k}\right),$$

from which we deduce the following identities:

$$\begin{cases} f(s_k)+\alpha &= -\dfrac{1}{s_k+1}+\circ\left(\dfrac{1}{k}\right)=-\dfrac{1}{\mathrm{i}k\pi}+\circ\left(\dfrac{1}{k}\right)\\ \frac{R-2}{R+2}f(s_k)+\alpha &= \left(1-\frac{R-2}{R+2}\right)\alpha+\circ(1)=8R/(R+2)^2+\circ(1)\,. \end{cases}$$

From Equation (6), we obtain

$$\frac{R-Z}{R+Z}=\frac{\frac{R-2}{R+2}f+\alpha}{f+\alpha},$$

and so, its value at point s_k is $-8\mathrm{i}k\pi R/(R+2)^2+\circ(k)$. Hence, we did exhibit (for any $R>2$) a sequence of points on the imaginary axis along which $(R-Z)/(R+Z)$ is unbounded. This shows it cannot lie in $\mathcal{H}^\infty$ (*cf.* Section 11).

Remark: $(R-Z)/(R+Z)$ does not belong either to the space $\mathcal{H}^2$ nor to any $\mathcal{H}^p$ (see definition in Section 11). To see this, remark that

$$f(s)=\frac{s^2\,\tanh(s)+s\,\tanh(s)-1}{s+1},$$

hence, setting $\beta=(R-2)/(R+2)$, we have

$$\frac{R-Z}{R+Z}=\frac{\beta f(s)+\alpha}{f(s)+\alpha}=\frac{\beta(s^2\,\tanh(s)+s\,\tanh(s)-1)+\alpha(s+1)}{s^2\,\tanh(s)+s\,\tanh(s)-1+\alpha\,(s+1)}.$$

Now, setting $M=1/\tanh(s)$ we can rewrite the previous expression as

$$\frac{R-Z}{R+Z}=\frac{\beta(1+1/s-M/s^2)+\alpha M/s+\alpha M/s^2}{1+(1+\alpha M)/s+(\alpha-1)M/s^2}.$$

Whenever $R>2$, we have $1<\alpha<2$ and $0<\beta<1$. Let us consider $s=\mathrm{i}\,(k\pi+t)$ where k is an integer such that $|k|\pi\geq 6+4/\beta$ and $t\in[\pi/4,3\pi/4]$. Then, $|s|\geq 6+4/\beta$ and $|M|\leq 1$. From this we deduce that

$$\left|\frac{R-Z}{R+Z}\right|\geq\frac{\beta\left(1-\frac{1}{6+4/\beta}-\frac{1}{(6+4/\beta)^2}\right)-\frac{2}{6+4/\beta}-\frac{2}{(6+4/\beta)^2}}{1+\frac{3}{6+4/\beta}+\frac{1}{(6+4/\beta)^2}}=\frac{29\beta^2+30\beta+8}{55\beta^2+60\beta+16}\,\beta\geq\frac{\beta}{2}.$$

In conclusion, we obtained a positive lower bound of $|(R-Z)/(R+Z)|$ valid on any interval of the form $[\mathrm{i}\pi(k+1/4),\mathrm{i}\pi(k+3/4)]$ ($k\in\mathbb{Z}$ large enough): this shows that $(R-Z)/(R+Z)$ does not belong to L^p of the imaginary axis, so it belongs to no Hardy space.

A pending issue. The example just constructed has pathological behavior because it has a sequence of poles located in the open left half-plane but asymptotically close to the imaginary axis at large frequencies. One may ask if a partial transfer function can be unstable (with respect to Definition 1) when the set of poles is at strictly positive distance from the axis. Symmetrically, if a partial transfer function has poles arbitrarily close to the axis, is it necessarily unstable? It is not known to the authors whether stability, in the sense of Definition 1, can be described solely in terms of poles of the partial transfer function. In this connection, we mention that the behavior of poles of rational functions in the variable s and in real powers of $\exp(s)$ (the subset of $\mathcal{E}$ attainable with lossless lines, *i.e.* those for which $a = c = 0$) is well-known [10]: they are asymptotic either to vertical lines or to curves separating exponentially fast from the imaginary axis. The situation with lossy lines has apparently not attracted much attention, and may be more complicated.

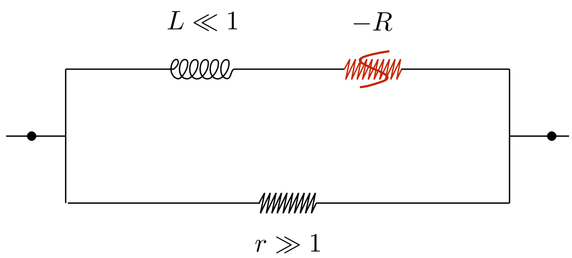
Until now, we considered linearized diodes as pure negative resistors and linearized transistors as pure current sources controlled by voltages. Such ideal models are simple and usually lead to good approximation at working frequencies, therefore they are widely used in simulation and design. However, the example of Figure 15 shows that checking stability does not reduce then to verify that poles have strictly negative real part. This somewhat contradicts common engineering practice, and puts a question mark on the use of ideal linearized models in connection with stability.

Actually, ideal models are somewhat unrealistic: even if it does not show at working frequencies, no active component has gain at all frequencies for there are always small resistive, capacitive and inductive effects in a physical device. As we will see in the next section, taking them into account restricts considerably the class of transfer functions realizable with such circuits.

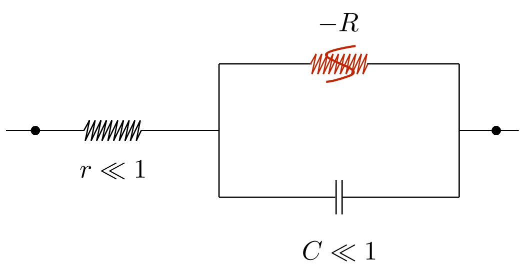
8 Realistic model of linearized components

So far, we modeled a diode as $i = f(u)$, where f is a non-linear real-valued function. To take into account small resistive, capacitive or inductive effects, we should rather postulate that $i = f(u, \mathrm{d}u/\mathrm{d}t, \mathrm{d}i/\mathrm{d}t)$ where f has very small derivatives with respect to the second and third variables.

Simple, "realistic" models of a diode can be given along these lines, *e.g.* those given in Figure 16a and Figure 16b.



(a) With inductive effect and high resistance (e.g., of the air around the diode)



(b) With capacitive effect and small resistance (e.g., of the wire)

Figure 16: Two realistic models of a linearized diode

Of course, physical reality is much more complex and it is pointless to attempt at giving a complete and accurate model of a linearized diode at all frequencies. Besides, different types of diodes would require different descriptions and it would be cumbersome to distinguish between them, while the differences are completely negligible in practice in the range of frequencies where the diodes are intended to be use. Instead, we put forward the paradigm that "what happens at very large frequencies is unimportant beyond passivity", and we shall use a somewhat abstract definition to accommodate various cases occurring in practice.

Definition 2. *A realistic linearized diode is a dipole with complex impedance Z having the following characteristics:*

- *$Z(s)$ is a rational function with real coefficients.*
- *When $|s| < \omega_0$, it holds that $Z(s) = -R + \epsilon(s)$ where ϵ is a function whose modulus is negligible compared to other quantities in the circuit.*
- *When $|s| > \omega_1$ and $\Re(s) \geq 0$, we have $\Re(Z(s)) \geq 0$.*

In this definition, $0 < \omega_0 < \omega_1$ as well as R are positive real numbers.

The hypothesis "Z is a rational function with real coefficients" simply states that a realistic negative resistor may, in principle, be described as a combination of a pure negative resistor and standard passive linear dipoles. In particular both models proposed in Figure 16 comply with Definition 2, but many other models would also meet our requirements.

Remark 5. *The same argument as in Remark 1 (*cf. *page 4) shows that, if V_1, V_2, I_1 and I_2 denote the potentials and currents at both terminals of a realistic linearized diode, then*

$$V_1\,\overline{I_1} + V_2\,\overline{I_2} = Z|I_1|^2 = Z|I_2|^2.$$

Therefore, when $|s| > \omega_1$ and $\Re(s) \geq 0$, we have that $\Re(V_1\,\overline{I_1} + V_2\,\overline{I_2}) \geq 0$.

Transistors can in turn be modeled in a "realistic" way, to account for the fact that they do not provide gain at very high frequencies. For instance, the model presented in Figure 17 corresponds to what is called the *intrinsic model of the linearized transistor* and displays capacitive effects appearing at the junctions between semiconductors. As in the case of a diode, reality is still more complex and involves both inductive and capacitive effects that we do not try to model since they are irrelevant to our discussion.

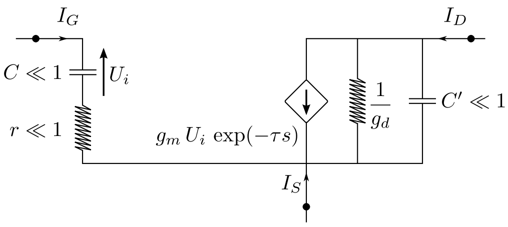


Figure 17: Intrinsic model of a linearized transistor

This leads us to the following definition, in the spirit of Definition 2:

Definition 3. *A realistic linearized transistor is an electronic component with three terminals such that voltages and currents, at these terminals, satisfy relations of the following form:*

$$\begin{pmatrix} -Y(s) & 0 & Y(s) & 1 & 0 & 0 \\ -Y_m(s) & -Y_d(s) & Y_m(s)+Y_d(s) & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} V_G \\ V_D \\ V_S \\ I_G \\ I_D \\ I_S \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where Y, Y_m and Y_d are complex admittances meeting the following requirements:

- *There exist positive real numbers $\tau_1, \dots, \tau_p$ such that Y, Y_m and Y_d are rational functions in the variables s, $\exp(-\tau_1 s)$, $\dots$, $\exp(-\tau_p s))$ having real coefficients.*
- *For $|s| < \omega_0$ it holds that $Y(s) = \epsilon(s)$, $Y_m(s) = g_m + \epsilon_m(s)$ and $Y_d(s) = g_d + \epsilon_d(s)$, where ϵ, ϵ_m and ϵ_d are functions whose moduli are negligible compared to other quantities in the circuit.*
- *The function $|Y_m(s)|$ tends to 0 as $|s|$ tends to infinity, subject to $\Re(s) \geq 0$. Moreover, whenever $|s|$ is large enough and $\Re(s) \geq 0$, it holds that $\Re(Y(s)) > \alpha$ and $\Re(Y_d(s)) > \beta$ with α and β two positive constants.*

Corollary 2. *For a realistic linearized transistor, it holds whenever $|s|$ is large enough with $\Re(s) \geq 0$ that*

$$\Re(V_G\,\overline{I_G} + V_D\,\overline{I_D} + V_S\,\overline{I_S}) \geq 0$$

Proof. From Definition 3, we see that, on the one hand, $I_G + I_D + I_S = 0$, and the other hand,

$$\begin{pmatrix} I_G \\ I_D \end{pmatrix} = \begin{pmatrix} Y(s) & 0 \\ Y_m(s) & Y_d(s) \end{pmatrix} \begin{pmatrix} V_G - V_S \\ V_D - V_S \end{pmatrix} = M\,V.$$

Now, $V_G\,\overline{I_G} + V_D\,\overline{I_D} + V_S\,\overline{I_S} = (V_G - V_S)\,\overline{I_G} + (V_D - V_S)\,\overline{I_D} + (V_S - V_S)\,\overline{I_S}$, since $I_G + I_D + I_S = 0$. Thus,

$$V_G\,\overline{I_G} + V_D\,\overline{I_D} + V_S\,\overline{I_S} = {}^tV\,\overline{M}\,\overline{V}.$$

We remark that ${}^tV\,\overline{M}\,\overline{V}$ is a 1×1 matrix, so it is equal to its transpose ${}^t\overline{V}\,M^\star\,V$. Consequently,

$$\Re(V_G\,\overline{I_G} + V_D\,\overline{I_D} + V_S\,\overline{I_S}) = \frac{1}{2}\left({}^t\overline{V}\,M^\star\,V + \overline{{}^tV\,\overline{M}\,\overline{V}}\right) = \frac{1}{2}\,{}^t\overline{V}\,(M^\star + M)\,V.$$

The hypotheses of Definition 3 ensure that the matrix $M + M^\star$ is Hermitian positive definite when $|s|$ is large enough and $\Re(s) \geq 0$, so $\Re(V_G\,\overline{I_G} + V_D\,\overline{I_D} + V_S\,\overline{I_S}) \geq 0$ for such s. ☐

It is immediate from Definitions 2 and 3 that partial frequency responses of circuits containing positive resistors, inductors, capacitors, transmission lines, and realistic linearized diodes and transistors are elements of the class $\mathcal{E}$ presented in Section 6. This comes from the fact that such a frequency response is still obtained by inverting a matrix whose entries are members of $\mathcal{E}$. However, not every function in $\mathcal{E}$ can be realized with a realistic circuit, in particular the function f from the ideal circuit in Figure 15 cannot. Indeed, this a consequence of the following theorem.

Theorem 3. *Let a circuit consists of resistors, inductors, capacitors, transmission lines and realistic linearized diodes and transistors (cf. Definitions 2 and 3). Assume as always that the behavior matrix of the circuit is invertible. To each partial frequency response $Z(s)$ of the circuit, there exists $K > 0$ such that, whenever $\Re(s) \geq 0$ and $|s| \geq K$, we have $\Re(Z(s)) \geq 0$.*

In order to prove this, recall Tellegen's theorem[3, thm. 2.14.1] expressing the conservation of power inside a circuit. For our purpose, it is convenient to state it in graph-theoretic form:

Theorem 4 (Tellegen)**.** *Let G be an oriented graph whose vertices are numbered from 1 to n and edges from 1 to m. Suppose that each vertex i is assigned a weight $V_i \in \mathbb{C}$ and that each edge j is assigned a weight $I_j \in \mathbb{C}$. Denote by $in(i)$ the set of incoming edges and by $out(i)$ the set of outgoing edges at vertex i. Moreover, for each edge k, we put $head(k)$ for the vertex k is pointing to and $tail(k)$ for the vertex k originates from. Subsequently, we set $U(k) = V_{tail(k)} - V_{head(k)}$.*

If G satisfies the junction rule

$$\forall i \in \{1, \dots n\}, \sum_{j \in in(i)} I_j = \sum_{j \in out(i)} I_j,$$

then the following relation holds:

$$\sum_j (V_{tail(j)} - V_{head(j)})\, I_j = 0. \tag{7}$$

Proof. Let us define the incidence matrix of G, say $M = (M_{ij})$, where

$$\begin{cases} M_{ij} = 1 \text{ if } i = \text{tail}(j), \text{ or, equivalently, if } j \in \text{out}(i), \\ M_{ij} = -1 \text{ if } i = \text{head}(j), \text{ or, equivalently, if } j \in \text{in}(i), \\ M_{ij} = 0 \text{ otherwise.} \end{cases}$$

The junction rule means that $MI = 0$. Moreover, if we let U indicate the vector with components $U(k)$, we get by definition ${}^tU = {}^tVM$. Consequently, ${}^tUI = {}^tVMI = 0$ which is (7). □

Remark 6. *If the junction rule is satisfied at all vertices but one, then it is satisfied at all vertices. To see it, observe that $(1, \dots, 1)\, M = (0, \dots, 0)$ because every column j of M contains exactly once the value 1 and once the value -1 (at rows corresponding, respectively, to the initial vertex and the final vertex of edge j). If the junction rule is satisfied at all vertices but one (we assume without loss of generality that it is vertex n), then the vector MI looks like ${}^t(0, \dots, 0, \alpha)$. Therefore, $\alpha = (1, \dots, 1)\, (MI)$. Finally, since $(1, \dots, 1)\, (MI) = ((1, \dots, 1)M)\, I = (0, \dots, 0)\, I = 0$, we conclude that α is indeed zero.*

We can now prove Theorem 3.

Proof of Theorem 3. We consider the situation described in Section 4.1. Thus, the circuit gets excited at some junction node α by a current source $i(t)$. Because the behavior matrix is invertible, voltages and currents are well defined in response to the excitation. We denote by I the Laplace transform of $t \mapsto i(t)$ and by $I_1, \dots, I_p$ and $V_1, \dots, V_n$ the Laplace transforms of currents and voltages in the circuit. We further fix the value of s so that $\Re(s) \geq 0$.

Define a graph G to be the circuit with one edge added between the ground and each transmission line (the virtual wire described in Figure 2), and we also add an edge between the ground and the junction node α. We set the weight of a junction node k to be $V_k(s)$. The weight of each node corresponding to an electronic component is defined as being 0. The weight of each edge j is $\overline{I_j(s)}$ (for the virtual wire of a transmission line, it is thus $-(\overline{I_{j'}(s)} + \overline{I_{j''}(s)})$ according to the last line of the matrix in Figure 2). The weight of the edge between the ground and node α is defined as $\overline{I(s)}$.

We first observe that G satisfies the junction rule. Indeed, for each vertex of G corresponding to a junction node of the circuit (except maybe the ground), this follows from Kirchhoff's law; at nodes that are electronic components, this is true by examination of the last line of the behavior matrix of each component (*cf.* Section 2). The junction rule is hence satisfied at all vertices of G except maybe for the ground. But from Remark 6, we see that the junction rule is then necessarily satisfied at all vertices including the ground. We can thus apply Tellegen's theorem.

By construction, all edges but the one between the ground and node α are oriented from a junction node to an electronic component. Since the weight of an electronic components is 0, each term $(V_{\mathrm{tail}(j)} - V_{\mathrm{head}(j)})\, I_j$ in Equation (7) becomes $V_{\mathrm{tail}(j)}(s)\,\overline{I_j(s)}$ in the present case. Moreover, since the ground has voltage 0, the term corresponding to the edge between the ground and node α is $(0 - V_\alpha(s))\,\overline{I(s)}$. Since by definition $V_\alpha(s) = Z(s)\,I(s)$, Tellegen's theorem, applied to the graph G, gives us

$$\sum_j V_{\mathrm{tail}(j)}(s)\,\overline{I_j(s)} = Z(s)\,|I(s)|^2. \tag{8}$$

Now, since each edge in the circuit is adjacent to exactly one electronic component, we can rewrite the left hand side of (8) as a sum over all electronic components i to obtain

$$\sum_i \sum_{\substack{\text{junction node } \beta \\ \text{adjacent to } i}} V_\beta(s)\,\overline{I_{\beta\to i}(s)} = Z(s)\,|I(s)|^2,$$

where $\overline{I_{\beta\to i}(s)}$ denotes the weight of the edge between node β and electronic component i. Applying Remark 1, Remark 5, Corollary 2 or Lemma 12 (*cf.* Section 10) according whether i is a passive dipole, a realistic linearized diode, a realistic linearized transistor or a transmission line, we see whenever $\Re(s) \geq 0$ and $|s|$ is large enough that, for each i,

$$\Re\left(\sum_\beta V_\beta(s)\,\overline{I_{\beta\to i}(s)}\right) \geq 0.$$

Hence, the left hand side of Equation (8) has positive real part provided that $\Re(s) \geq 0$ and $|s|$ is large enough, and so does $Z(s)$. □

Below we draw two important consequences of Theorem 3. They point at a remarkable difference between the case of ideal linearized elements described in Section 6, and the case of realistic linearized elements.

Corollary 3. *Let Z be a partial frequency response of a circuit made of transmission lines, resistors, capacitors, inductors, and realistic linearized diodes and transistors. For $\Re(s) \geq 0$ and $|s|$ large enough, it holds for all $R \geq 0$ that $|(R - Z(s))/(R + Z(s))| \leq 1$.*

Proof. Let K be as in Theorem 3. Then, for $|s| > K$ and $\Re(s) \geq 0$, we have as soon as $R \geq 0$ that $|\Re(R - Z(s))| \leq \Re(R + Z(s))$, and it is otherwise clear that $\Im(R - Z(s)) = -\Im(R + Z(s))$, hence $|R - Z(s)| \leq |R + Z(s)|$, as desired. □

Corollary 4. *Assumptions and notations being as in Theorem 3, it holds for each $R \geq 0$ that the meromorphic function $(R - Z)/(R + Z)$ has finitely many poles in the closed right half plane.*

Proof. From the previous corollary we see that the poles of $(R-Z)/(R+Z)$ in the closed right half plane have to lie in a disk of radius K centered at 0. Since $(R-Z)/(R+Z) \in \mathcal{E}$, it is a meromorphic function on $\mathbb{C}$ by Remark 4. Hence poles cannot accumulate, therefore they are finite in number. □

9 A stability criterion

Below we set up a criterion for stability, in the sense of Definition 1, of a partial frequency response of a circuit made of transmission lines, resistors, capacitors, inductors, and realistic linearized diodes and transistors. It turns out to be similar to the classical passivity criterion [5, ch.6] , except that the condition "residues of imaginary poles should be positive" gets relaxed into "residues of imaginary poles should have positive real part".

Theorem 5. *Let Z be a partial frequency response of a circuit made of transmission lines, resistors, capacitors, inductors, and realistic linearized diodes and transistors. Then, Z is stable in the sense of Definition 1 if and only if it has no pole in the open right half plane, while each pole it may have on the imaginary axis is simple and has a residue with strictly positive real part.*

Proof. From Corollary 3 we know there is $K > 0$ such that $|(R - Z(s))/(R + Z(s))| \leq 1$ as soon as $\Re(s) \geq 0$ and $|s| > K$, therefore the stability of Z in the sense of Definition 1 is equivalent to the boundedness of $(R-Z)/(R+Z)$ for $|s| \leq K$ and $\Re s > 0$, uniformly with respect to large R.

Let ζ be a pole of Z with $|\zeta| \leq K$, and $m \geq 1$ be its multiplicity. In a neighborhood of ζ, we can write

$$Z(s) = a_m(s-\zeta)^{-m} + \cdots + a_1(s-\zeta)^{-1} + F(s), \quad a_m \neq 0, \tag{9}$$

where $F(s)$ is holomorphic and bounded in some disk $D(\zeta, a) = \{s : |s-\zeta| < a\}$. By (9), Z is holomorphic from $D(\zeta, a)$ into the Riemann sphere, and as such it is an open map, meaning that the image of an open set is open. In particular, the image of $D(\zeta, a)$ under Z is a neighborhood of ∞, meaning that it contains $\{s : |s| > A\}$ for some A.

If $\Re\zeta > 0$, we can pick a so small that $D(\zeta, a)$ is included in the open right half plane, and by what precedes Z takes every negative value of sufficiently large modulus in $D(\zeta, a)$. Hence Z cannot be stable, because $(R-Z)/(R+Z)$ will have a pole nearby ζ for $R > 0$ large enough (note that $R - Z$ cannot vanish at a point where $Z = -R$).

Assume now that $\Re\zeta = 0$. To say that $|(R-Z)/(R+Z)| \leq M$ for some $M > 0$ is equivalent to say that $|1 + R/Z| \geq \varepsilon_1$ for some $\varepsilon_1 > 0$. Now, if $s \in D(\zeta, a)$, then by (9)

$$1 + \frac{R}{Z(s)} = \frac{a_m + a_{m-1}(s-\zeta) + \cdots + a_1(s-\zeta)^{m-1} + (R + F(s))(s-\zeta)^m}{a_m + a_{m-1}(s-\zeta) + \cdots + a_1(s-\zeta)^{m-1} + F(s)(s-\zeta)^m}. \tag{10}$$

If $m \geq 2$, then to each $R > 0$ large enough and each $\varepsilon > 0$, one can find $s_{R,\varepsilon} \in D(\zeta, a)$, with $\Re s_{R,\varepsilon} > 0$, such that

$$|(R + F(\zeta))(s_{R,\varepsilon} - \zeta)^m + a_m| < \varepsilon. \tag{11}$$

Indeed, the image of $D(a, \zeta)$ under the map $s \mapsto (s-\zeta)^m$ is $D(0, a^m)$ if $m \geq 3$, and $D(0, a^2)$ deprived from the negative real axis if $m = 2$. Moreover, it is clear that $s_{R,\varepsilon}$ has to converge to ζ when $R \to \infty$, uniformly with respect to ε. Therefore, if we let $\{R_k\}$ be a sequence tending to $+\infty$, we readily see from (10) that $|1 + R/Z(s_{R_k,1/k})|$ goes to zero as $k \to \infty$ so that $(R-Z)/(R+Z)$ cannot be uniformly bounded with respect to R in $D(\zeta, a) \subset \{s : \Re s > 0\}$. Hence Z is unstable.

Finally if $m = 1$ and R is large enough, it is easily checked that there exists $s_{R,\varepsilon} \in D(\zeta, a)$ satisfying $\Re s_{R,\varepsilon} > 0$ and such that (11) holds for arbitrary small ε if and only if $\Re a_1 \leq 0$. This establishes the only if part of Theorem 5.

Conversely, suppose that $Z(s)$ has only pure imaginary poles for $\Re s \geq 0$ and $|s| \leq K$, which are simple and whose residue has strictly positive real part. Let ζ be such a pole and a_1 its residue. We just saw that $s_{R,\varepsilon}$ as in (11) does not exist for large R and small ε, therefore there is $\varepsilon_0 > 0$ such that $|(R + F(\zeta))(s - \zeta) + a_1| \geq \varepsilon_0$ for $s \in D(\zeta, a)$ and all R large enough. Then, by inspection of (10), we see that $|1 + R/Z(s)| \geq \varepsilon_0/(2|a_1|)$ as soon as R is large enough and $s - \zeta$ is small enough, $\Re s > 0$. That is to say, there is $r_0 > 0$ and $R_0 > 0$ such that $|s - \zeta| < r_0$, $\Re s > 0$, ad $R > R_0$ together imply $|(R - Z(s))/(R + Z(s))| \leq M$ for some M independent of R and s satisfying the preceding conditions.

Because there are only finitely many poles, the preceding argument shows that we can choose $R_0 > 0$ and $M > 0$ so large, and $r_0 > 0$ so small, that $|(R - Z(s))/(R + Z(s))| \leq M$ as soon as s lies in the right half plane but closer than r_0 to one of the poles. But if $\Re s > 0$, $|s| \leq K$, and the distance from s to one of the poles is bigger than r_0, then $Z(s)$ is bounded independently of s and increasing R_0 if necessary we may assume that still $|(R - Z(s))/(R + Z(s))| \leq M$ for such s and $R > R_0$ (note that $(R - Z(s))/(R + Z(s))$ tends to 1 as R tends to $+\infty$ while $Z(s)$ remains bounded). This shows that Z is stable, as announced. □

10 Appendix 1: Telegrapher's equation

A transmission line is commonly modeled as a succession of infinitesimal capacitors, resistors and inductors whose impedances do not depend on their position on the line (see Figure 18. In the figure, G denotes the conductance of the resistor).

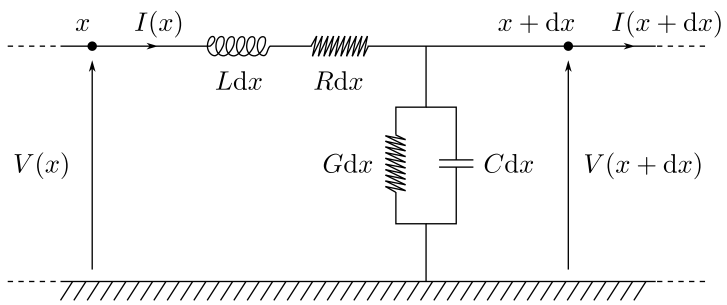


Figure 18: Model of a transmission line (the line goes from $x = 0$ to $x = \ell$ and is a succession of the same LR-GC infinitesimal elements)

This model leads to what is known as Telegrapher's equation: expressing the relation between currents and voltages at positions x and $x + \mathrm{d}x$, one sees that

$$\begin{cases} V(x + \mathrm{d}x) - V(x) = -(R + Ls)I(x)\,\mathrm{d}x \\ V(x + \mathrm{d}x) = \dfrac{1}{(G + Cs)\mathrm{d}x}\,(I(x) - I(x + \mathrm{d}x)). \end{cases}$$

From these local equations, we deduce a system of first order equations:

$$\begin{cases} \dfrac{\partial V}{\partial x} = -(R + Ls)I \\ \dfrac{\partial I}{\partial x} = -(G + Cs)V \end{cases} \Longrightarrow \begin{cases} \dfrac{\partial^2 V}{\partial x^2} = \gamma^2\, V \\ \dfrac{\partial^2 I}{\partial x^2} = \gamma^2 I, \end{cases} \tag{12}$$

where γ is one of the complex square roots of $(R + Ls)(G + Cs)$.

We set $z_0 = (R + Ls)/\gamma$, so that we can write $R + Ls = \gamma\, z_0$ and $G + Cs = \dfrac{\gamma}{z_0}$.

Since R, G, L and C do not depend on x, Equation (12) leads to the following explicit solutions:

$$\begin{cases} V(x) = A \exp(\gamma\, x) + B \exp(-\gamma\, x) \\ I(x) = D \exp(\gamma\, x) + E \exp(-\gamma\, x). \end{cases} \tag{13}$$

The transmission line has a given length ℓ and we want to express the relations between $I(0)$, $V(0)$, $I(\ell)$ and $V(\ell)$. For this purpose, we simply need to express the constants A, B, D and E in function of $I(0)$ and $V(0)$.

On the one hand, $V(0) = A + B$ and $I(0) = D + E$ by letting $x = 0$ in Equation (13). On the other hand, differentiating Equation (13) at point $x = 0$, we get

$$\begin{cases} \dfrac{\partial V}{\partial x}(0) = \gamma\,(A - B) \\ \dfrac{\partial I}{\partial x}(0) = \gamma\,(D - E). \end{cases} \tag{14}$$

Now, using Equation (12) at $x = 0$, we see that $\dfrac{\partial V}{\partial x}(0) = -\gamma\, z_0\, I(0)$ and $\dfrac{\partial I}{\partial x}(0) = -\dfrac{\gamma}{z_0}\, V(0)$, thus $B - A = z_0\, I(0)$ and $E - D = V(0)/z_0$. This gives us the values of A, B, D and E:

$$\begin{cases} A = \frac{1}{2}\,(V(0) - z_0\, I(0)) \\ B = \frac{1}{2}\,(V(0) + z_0\, I(0)) \\ D = \frac{1}{2}\,(I(0) - V(0)/z_0) \\ E = \frac{1}{2}\,(I(0) + V(0)/z_0). \end{cases} \tag{15}$$

Finally, putting these constants in Equation (13), at point $x = \ell$, and collecting the terms in $I(0)$ and in $V(0)$, we obtain:

$$\begin{cases} V(\ell) &= \cosh(\gamma\,\ell)\, V(0) &- \; z_0 \sinh(\gamma\,\ell)\, I(0) \\ I(\ell) &= \dfrac{-\sinh(\gamma\,\ell)}{z_0}\, V(0) &+ \; \cosh(\gamma\,\ell)\, I(0). \end{cases} \tag{16}$$

The second line of Equation (16) gives us

$$V(0) = z_0 \coth(\gamma\,\ell)\, I(0) - \frac{z_0}{\sinh(\gamma\,\ell)}\, I(\ell),$$

which we inject in the first line of the system, leading to

$$V(\ell) = \frac{z_0}{\sinh(\gamma\,\ell)}\, I(0) - z_0 \coth(\gamma\,\ell)\, I(\ell).$$

Remark that multiplying R, G, L and C by some constant α does not change the value of z_0, but has the effect of multiplying γ by α. Therefore, the relations between the currents and voltages at the terminals of a line of length ℓ with characteristics R, G, L and C are the same that the relations at the terminals of a line of length 1 with characteristics R/ℓ, G/ℓ, L/ℓ and

C/ℓ. In conclusion, from a theoretical viewpoint, the length of the line can be arbitrary and is not worth mentioning. We arbitrarily set it to 1 in this report.

Note that, by convention, all currents are oriented so as to enter electronic components. In Figure 2 on page 5, we hence have $I_1 = I(0)$, but $I_2 = -I(\ell)$.

Lemma 12. *When $\Re(s) \geq 0$, we have $\Re(V_1\,\bar{I}_1 + V_2\,\bar{I}_2) \geq 0$.*

Proof. We remark that

$$\begin{aligned} V_1\,\bar{I}_1 + V_2\,\bar{I}_2 &= V(0)\,\overline{I(0)} - V(\ell)\,\overline{I(\ell)} \\ &= -\int_0^\ell \left(\frac{\partial V}{\partial x}(\xi)\,\overline{I(\xi)} + V(\xi)\,\frac{\partial \bar{I}}{\partial x}(\xi)\right)\,\mathrm{d}\xi. \end{aligned}$$

Replacing $\partial V/\partial x$ and $\partial I/\partial x$ by their values in function of I and V, as given by Equation (12), we obtain

$$V_1\,\bar{I}_1 + V_2\,\bar{I}_2 = -\int_0^\ell \left(-(R+Ls)\,|I(\xi)|^2 - \overline{(G+Cs)}\,|V(\xi)|^2\right)\,\mathrm{d}\xi.$$

Finally, we get that

$$\Re(V_1\,\bar{I}_1 + V_2\,\bar{I}_2) = \int_0^\ell (R + L\,\Re(s))\,\,|I(\xi)|^2 + (G + C\,\Re(\bar{s}))\,\,|V(\xi)|^2\,\mathrm{d}\xi.$$

The expression under the integral symbol being positive for $\Re(s) \geq 0$, the integral itself is positive when $\Re(s) \geq 0$. ☐

11 Appendix 2: Transfer functions and stability

In this section, we discuss transfer functions in connection with stability and harmonic response, a thorough account of which seems hard to find in the literature.

On the real line $\mathbb{R}$, we let $L^1(\mathbb{R})$, $L^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$ indicate respectively the spaces of complex-valued summable, square summable, and essentially bounded measurable functions with respective norms

$$\|f\|_{L^1(\mathbb{R})} = \int_{-\infty}^{+\infty} |f(t)|\,dt, \qquad \|f\|_{L^2(\mathbb{R})} = \left(\int_{-\infty}^{+\infty} |f(t)|^2\,dt\right)^{1/2} \tag{17}$$

and

$$\|f\|_{L^\infty(\mathbb{R})} = \sup\left\{A \geq 0 :\; m\big(\{x \in \mathbb{R};\; |f(x)| > A\}\big) > 0\right\}, \tag{18}$$

where $m(E)$ stands for Lebesgue measure of a set E. We put $L^1_{loc}(\mathbb{R})$ (resp. $L^2_{loc}(\mathbb{R})$, $L^\infty_{loc}(\mathbb{R})$) for spaces of functions f such that $\chi_E f \in L^1(\mathbb{R})$ (resp. $L^2(\mathbb{R})$, $L^\infty(\mathbb{R})$) whenever E is a bounded measurable subset of $\mathbb{R}$, where χ_E stands for the characteristic function of E (which is 1 on E and 0 elsewhere). We shall have to deal with corresponding spaces when $\mathbb{R}$ gets replaced by the imaginary axis $i\mathbb{R}$, or by the positive semi-axis $\mathbb{R}^+ = [0, \infty)$. We then write $L^1(i\mathbb{R})$, $L^1(\mathbb{R}^+)$, and so on.

In order to study even fairly common systems[1], one cannot work entirely with functions and it is convenient to use the distributional formalism as follows. Let $\mathcal{D}(\mathbb{R})$ be the space of smooth (*i.e.* C^∞) functions with compact support on $\mathbb{R}$; recall that the support of φ, abbreviated as

[1] For instance transmission lines like in the previous Appendix.

$\operatorname{supp}\varphi$, is the closure of those $t \in \mathbb{R}$ with $\varphi(t) \neq 0$. A distribution [12, ch. 6] is a form (*i.e.* a complex-valued linear map) Φ on $\mathcal{D}(\mathbb{R})$ which is continuous in the sense that to each compact $K \subset \mathbb{R}$, there is a constant C and an integer $n \geq 0$ for which $|\langle \Phi, \varphi\rangle| \leq C\|\varphi^{(k)}\|_{L^\infty(\mathbb{R})}$ for all $0 \leq k \leq n$, whenever $\varphi \in \mathcal{D}(\mathbb{R})$ is supported on K; here and below, brackets indicate the action of distributions and superscript (k) stands for the k-th derivative. Each $f \in L^1_{loc}(\mathbb{R})$ identifies with a distribution upon setting $\langle f, \varphi\rangle = \int_{\mathbb{R}} f\varphi$. Distributions can be multiplied by smooth functions: if $\psi \in C^\infty(\mathbb{R})$, then $\psi\Phi$ is the distribution given by $\langle \psi\Phi, \varphi\rangle = \langle \Phi, \psi\varphi\rangle$. They can be differentiated as well, the derivative of Φ being given by $\langle \Phi^{(1)}, \varphi\rangle = -\langle \Phi, \varphi^{(1)}\rangle$. Also, if Φ is a distribution and ψ lies in $\mathcal{D}(\mathbb{R})$, the convolution $\Phi * \psi$ is the function $\Phi * \psi(t) = \langle \Phi, \psi(t - .)\,\rangle$; here, the dot in the argument stands for a dummy variable. The support $\operatorname{supp}\Phi$ of a distribution Φ is the complement of the largest open set Ω for which $\langle \Phi, \varphi\rangle = 0$ whenever $\operatorname{supp}\varphi \subset \Omega$.

Let us think of $t \in \mathbb{R}$ as being time. A linear dynamical system is a linear map $u \mapsto y$ sending each u belonging to a certain function space of the variable t (the admissible inputs) to a function y of t (the output corresponding to u), which is causal (*i.e.* $u(t) = 0$ for $t \leq t_0$ implies the same is true for y) and time invariant (if y is the output corresponding to some admissible u and if $\tau \in \mathbb{R}$, then $u(. - \tau)$ is admissible and $y(. - \tau)$ is the corresponding output). If inputs from $\mathcal{D}(\mathbb{R})$ are admissible and generate continuous outputs, then under mild continuity assumptions[2] there is a distribution Φ supported on $\mathbb{R}^+$ allowing us to describe the system through the convolution equation:

$$y(t) = (\Phi * u)(t), \qquad u \in \mathcal{D}(\mathbb{R}). \tag{19}$$

Moreover inputs from $\mathcal{D}(\mathbb{R})$ turn out to generate smooth outputs [12, thm. 6.33]. The distribution Φ is called the impulse response of the system as it corresponds formally to the output generated by a Dirac-δ input. What we defined really is a scalar system, that is, one whose input and output are real or complex valued. The more general case of vector-valued (finite dimensional) input and output can be represented as a matrix of scalar systems. For matters under consideration here, results for vector-valued inputs and outputs are immediately deduced by concatenation from their scalar-valued analogs.

Let now $\mathcal{S}(\mathbb{R})$ be the Schwartz space of smooth functions whose derivatives of any order decrease faster than every polynomial at infinity. A distribution Ψ is called tempered if it extends to a form on $\mathcal{S}$ which is continuous in that there are integers $n, m \geq 0$ and a constant C for which $|\langle \Psi, \varphi\rangle| \leq C\|(1 + |.|^m)\varphi^{(k)}\|_{L^\infty(\mathbb{R})}$ for all $0 \leq k \leq n$, whenever $\varphi \in \mathcal{S}(\mathbb{R})$. Now, if Φ is a distribution supported on $\mathbb{R}^+$ such that the distribution $e^{-A\cdot}\,\Phi$ is tempered for some $A \in \mathbb{R}$, then we say that Φ is Laplace transformable (or A-Laplace transformable if some A needs to be specified) and we can define its Laplace transform $\mathcal{L}(\Phi)$ as a function of the complex variable $s = x + iy$ defined on the semi-infinite strip $x > A$ by the rule:

$$\mathcal{L}(\Phi)(s) = \langle \Phi, e^{-s\cdot}\rangle. \tag{20}$$

Notation (20) deserves a word of explanation: although Φ cannot be applied, strictly speaking, to the function on $\mathbb{R}^+$ given by $t \mapsto e^{-st}$, $\Re s > A$, the latter can be smoothly extended for negative t so as to vanish, say for $t \leq -1$. The tempered distribution $e^{-A\cdot}\,\Phi$ may then be applied to $e^{A\cdot}$ times this extension which lies in $\mathcal{S}(\mathbb{R})$, and the fact that Φ is supported on $\mathbb{R}^+$ entails that the result is independent of the extension used. This is what is meant by (20). It can be shown that the latter defines a holomorphic function of s for $\Re s > A$ [9, thm. 8.3.1]. If Φ happens to be a locally integrable function, say $f \in L^1_{loc}(\mathbb{R})$, then it is easy to see that it is A-Laplace transformable if and only if $e^{-A\cdot} f \in L^1(\mathbb{R})$ and that (20) reduces to the standard definition of

[2] If $\{u_n\} \subset \mathcal{D}(\mathbb{R})$ is supported on a compact set K and $u_n^{(k)}$ converges to $u^{(k)}$ uniformly on K for each $k \geq 0$ and some $u \in \mathcal{D}(\mathbb{R})$, the corresponding outputs $\{y_n\}$ should converge pointwise to the output associated with u.

Laplace transform:

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\, dt. \tag{21}$$

Note for later use that if Φ is Laplace transformable, then its derivative is again Laplace transformable and $\mathcal{L}(\Phi^{(1)})(s) = s\mathcal{L}(\Phi)(s)$. This follows easily from the definition.

When Φ is A-Laplace transformable and $u \in \mathcal{D}(\mathbb{R})$, then y given by (19) is in turn A'-Laplace transformable for all $A' > A$. Indeed, it holds that

$$\begin{aligned} e^{-A't}|y(t)| &= \left|e^{-A't}\langle \Phi, u(t-.)\rangle\right| = e^{-(A'-A)t}\left|\langle e^{-A\cdot}\Phi, e^{-A(t-.)}u(t-.)\rangle\right| && (22)\\ &\leq e^{-(A'-A)t} C \left\|(1+|.|^m)\left(e^{-A\cdot}u\right)^{(n)}\right\|_{L^\infty(\mathbb{R})} = C' e^{-(A'-A)t}, && (23)\end{aligned}$$

where we used continuity properties of tempered distributions. Thus, since Laplace transform converts convolutions into ordinary products much like Fourier transform does [9, eqn. 8.5.8], we get that

$$\mathcal{L}(y)(s) = \mathcal{L}(\Phi)(s)\mathcal{L}(u)(s), \qquad \Re s > A. \tag{24}$$

Equation (24) is the so-called frequency description of (19), and $\mathcal{L}(\Phi)$ is called the transfer function of the system. Note that the restriction of $\mathcal{L}(y)$ to the vertical line $\{\Re s = x > A\}$ is the Fourier transform of $e^{-x\cdot}y \in L^1(\mathbb{R})$, hence $\mathcal{L}(y)$ cannot vanish identically unless y does [11, thm. 9.11]. Consequently, by (24), a system is uniquely defined by its transfer-function when it exists.

Although (19) is initially valid for $u \in \mathcal{D}(\mathbb{R})$ only, the class of admissible inputs is much bigger in cases of interest and (24) will usually extend to more general functions. For instance if continuous inputs with compact support are admissible and generate continuous outputs, and if moreover bounded pointwise convergence of inputs entails pointwise convergence of outputs, then by the Riesz representation theorem [11, thm. 6.19] the impulse response Φ is locally a finite measure, say ν, and (19) becomes

$$y(t) = \int_{[0,t]} u(t-\tau)\, d\nu(\tau) \tag{25}$$

(compare [1, Ch. 1]). Letting $|\nu|$ be the total variation of ν, which is a (not necessarily finite) positive measure on $\mathbb{R}^+$ [11, ch. 6, sec.1], we find that Φ is A-Laplace transformable if and only if $t \mapsto e^{-At}$ is summable against $|\nu|$. In this case (24) will hold in some strip $\Re s > A'$ as soon as $u \in L^1_{loc}(\mathbb{R})$ is of exponential type, meaning there is $B \in \mathbb{R}$ such that $e^{-B\cdot}u \in L^\infty(\mathbb{R})$. This level of generality is about right for most applications to circuit theory, for solutions to linear differential equations with constant coefficients initially at rest can be expressed *via* convolution against exponential kernels while delays introduced by transmission lines introduce Dirac delta measures in the impulse response [2, 3].

It has become customary to denote signals (*i.e.* functions of t) by lower case letters and their Laplace transforms with corresponding upper case letters. Real-valued signals $t \mapsto u(t)$ correspond under Laplace transform to conjugate-symmetric functions: $U(\bar{s}) = \overline{U(s)}$. System (19) maps real signals to real signals if and only if Φ is real-valued on real functions, which results in the transfer function being conjugate symmetric. System theory is mostly concerned with real-valued signals although such a restriction is rather immaterial as far as theory goes.

What precedes is enough to bring formal meaning to most computations, as long as pointwise evaluation in (24) is restricted to some strip where every Laplace transform involved does exist. However, when dealing with issues of passivity and stability, square summable inputs and outputs are of special significance (these are the signals of finite energy) and a more refined interpretation of (24), valid *on* the imaginary axis, becomes necessary.

Clearly if $f \in L^2(\mathbb{R}^+)$, then it is A-Laplace transformable for each $A > 0$ because $f(t)e^{-at}$ is summable by the Schwarz inequality. Its Laplace transform F is thus defined and holomorphic for $\Re s > 0$ and, as can be surmised from (21) and rigorously proved [7, ch. 8], the limit $F(iy) = \lim_{x\to 0} F(x+iy)$ exists for almost every $y \in \mathbb{R}$ and coincides with the Fourier transform $\hat{f}(y)$ defined for almost every $y \in \mathbb{R}$ by the formula

$$\hat{f}(y) = \lim_{B\to+\infty} \int_0^B e^{-iyt} f(t)\, dt. \tag{26}$$

The limit in (26) holds in the L^2-sense by Plancherel's theorem [11, thm. 9.13]. The fact that it also holds pointwise almost everywhere is a deep result after the work of Carleson [6, thm. 11.1.1]. Furthermore, putting $F_x(y) = F(x+iy)$, we have that $\lim_{x\to 0} \|F_x - \hat{f}\|_{L^2(\mathbb{R})} = 0$ and also that

$$\sup_{x>0} \int_{-\infty}^{+\infty} |F(x+iy)|^2\, dy < \infty. \tag{27}$$

In fact the supremum in (27) is $\|\hat{f}\|^2_{L^2(\mathbb{R})}$, which is in turn equal to $2\pi\|f\|^2_{L^2(\mathbb{R}^+)}$ by the essentially isometric character of the Fourier transform. It is a theorem of Paley and Wiener that the space of all functions F holomorphic in the right half-plane and satisfying (27) coincides with the set of Laplace transforms of functions $f \in L^2(\mathbb{R}^+)$. When normed with the square root of the left hand side of (27), the latter becomes a Hilbert space known as the *Hardy space* of exponent 2 of the right half plane [7, Ch. 8][3], hereafter denoted by by $\mathcal{H}^2$. The norm of a Hardy function $F \in \mathcal{H}^2$, indicated with $\|F\|_2$, is thus $\sqrt{2\pi}$ times the energy of the signal of which it is the Laplace transform. Conjugate symmetric $\mathcal{H}^2$-functions. form a *real* Hilbert subspace of $\mathcal{H}^2$.

We need also introduce the Hardy space $\mathcal{H}^\infty$ of bounded analytic functions in $\{\Re s > 0\}$ endowed with the *sup* norm. As is the case for $\mathcal{H}^2$ functions, the limit $F(iy) = \lim_{x\to 0} F(x+iy)$ exists for almost every $y \in \mathbb{R}$ when $F \in \mathcal{H}^\infty$ [7, ch. 8]. Moreover, this limit function lies in $L^\infty(i\mathbb{R})$, and it holds that

$$\|F\|_{L^\infty(i\mathbb{R})} = \sup_{\Re s>0} |F(s)|. \tag{28}$$

The quantity in (28) is the $\mathcal{H}^\infty$-norm of F that we abbreviate as $\|F\|_\infty$.

Consider now a linear dynamical system mapping $L^2(\mathbb{R}^+)$ into itself, that is, the system maps inputs of finite energy to outputs of finite energy Such a system we call *stable*. It is remarkable that it is automatically continuous as a map, that is, there is a constant $C \geq 0$ such that $\|y\|_{L^2(\mathbb{R}^+)} \leq C\|u\|_{L^2(\mathbb{R}^+)}$ [10, thm. 4.1.1]. Taking Laplace transforms, we find that the system induces a continuous linear map from $\mathcal{H}^2$ into itself that commutes with multiplication by $e^{-s\tau}$ for all $\tau \in \mathbb{R}^+$. As it turns out [10, Cor. 3.2.4], such a linear map is just multiplication by some $H \in \mathcal{H}^\infty$, and its norm is $\|H\|_\infty$. Next, pick $a < 0$ and consider the function $G(s) = H(s)/(s-a)$. Because $H \in \mathcal{H}^\infty$, it is easy to check that $G \in \mathcal{H}^2$ hence by the Paley Wiener theorem there is $g \in L^2(\mathbb{R}^+)$ such that $G = \mathcal{L}(g)$. If we regard g as a distribution, we find that $H = \mathcal{L}(g^{(1)} - ag)$. Thus, since a system is uniquely defined by its transfer function, we conclude that our initial system has Laplace transformable impulse response $g^{(1)} - ag$ and transfer function H. To recap, a system is stable if and only if it has a transfer function lying in $\mathcal{H}^\infty$, and the maximum gain is the $\mathcal{H}^\infty$-norm of the transfer function. We also gather from what precedes that the impulse response is of the form $g^{(1)} - ag$ with $g \in L^2(\mathbb{R}^+)$, $a < 0$, and that every function in $\mathcal{H}^\infty$ arises

[3] More generally, one defines the Hardy space $\mathcal{H}^p$, $1 \leq p < \infty$, to consist of holomorphic functions in the right half plane such that $\sup_{x>0} \|F(x+\,.)\|_{L^p(i\mathbb{R})} < \infty$., see [7, ch.8]

as the transfer function of a stable system. However, not every $g \in L^2(\mathbb{R}^+)$ gives rise to the impulse response of a stable system *via* $\Phi = g^{(1)} - ag$ for some $a < 0$. In fact, no non-tautological characterization is known of those Laplace transformable distributions whose Laplace transform belongs to $\mathcal{H}^\infty$, compare [9, thm. 8.7.1].

A stable system can be fed with an input $u \in L^\infty_{loc}(\mathbb{R}^+)$ (extended by 0 for negative t) because, by causality, the output at time $t \le t_0$ is the same as if we used the input $\chi_{[0,t_0]}u$ which lies in $L^2(\mathbb{R}^+)$. In order to gain physical understanding of the transfer function, it is worth estimating the asymptotic behavior, as $t \to +\infty$, of the output generated by the input $u(t) = \chi_{\mathbb{R}^+}(t)e^{i\omega t}$ for fixed $\omega \in \mathbb{R}$. Dwelling on what precedes, let the system have transfer function $H \in \mathcal{H}^\infty$ and impulse response $\Phi = g^{(1)} - ag$ with $g \in L^2(\mathbb{R})$, $a < 0$. Due to the distributional nature of the derivative $g^{(1)}$ appearing in Φ, we cannot bluntly plug in (19) the input u which is non-smooth at zero and has infinite support. To circumvent this, let $\varphi \in \mathcal{D}(\mathbb{R})$ be non-negative, supported on $[-1, 1]$, even, and such that $\int \varphi(t)\,dt = 1$. For $\varepsilon \in (0, 1]$, set $\varphi_\varepsilon(t) = \varphi(t/\varepsilon)/\varepsilon$ and $\phi_\varepsilon(t) = \int_{-1}^t \varphi_\varepsilon(\tau)\,d\tau$. Pick $T > 2$, define $\phi_{\varepsilon,T}(t) = \phi_\varepsilon(t)\phi_\varepsilon(T - t)$, and put $u_{\varepsilon,T}(t) = \phi_{\varepsilon,T}(t)e^{i\omega t}$. By construction $u_{\varepsilon,T} \in \mathcal{D}(\mathbb{R})$, hence using (19) we obtain the corresponding output $y_{\varepsilon,T}$ by the formula:

$$
\begin{aligned}
y_{\varepsilon,T}(t) &= e^{i\omega t} \int_{\mathbb{R}} g(\tau) \left(\phi^{(1)}_{\varepsilon,T}(t-\tau) + (i\omega - a)\phi_{\varepsilon,T}(t-\tau) \right) e^{-i\omega\tau}\,d\tau \qquad (29)\\
&= e^{i\omega t}\Big(\big(ge^{-i\omega\cdot} * \varphi_\varepsilon\big)(t) - \big(ge^{-i\omega\cdot} * \varphi_\varepsilon(T - .)\big)(t) + (i\omega - a)\big(ge^{-i\omega\cdot} * \phi_{\varepsilon,T}\big)(t)\Big).
\end{aligned}
$$

When ε tends to 0, then $u_{\varepsilon,T}$ converges in $L^2(\mathbb{R})$ to $u_T = \chi_{[0,T]}e^{i\omega\cdot}$ so that $y_{\varepsilon,T}$ must converge to the corresponding output y_T, as the system is stable. Now, since $ge^{-i\omega\cdot} \in L^2(\mathbb{R})$ (when extended by zero for negative arguments), it is standard that $ge^{-i\omega\cdot} * \varphi_\varepsilon \to ge^{-i\omega\cdot}$ in $L^2(\mathbb{R})$ [13, thm. 1.6.1] and by the same token we get that $ge^{-i\omega\cdot} * \varphi_\varepsilon(. - T) \to g(. - T)e^{-i\omega\,(.-T)}$. Moreover, because $\phi_{\varepsilon,T}$ converges to $\chi_{[0,T]}$ in $L^1(\mathbb{R})$, it follows from Minkowski's inequality for integrals [11, ch. 7, ex. 4] that $ge^{-i\omega\cdot} * \phi_{\varepsilon,T} \to ge^{-i\omega\cdot} * \chi_{[0,T]}$ in $L^2(\mathbb{R})$. Altogether, we find that

$$
y_T(t) = g(t) - g(t - T)e^{i\omega T} + e^{i\omega t}(i\omega - a) \int_{\max(0,t-T)}^t g(\tau)e^{-i\omega\tau}\,d\tau.
$$

Next, let y be the output associated with $u(t) = \chi_{\mathbb{R}^+}(t)e^{i\omega t}$. By causality it must coincide with y_T on $[0, T]$, therefore using that g is supported on $\mathbb{R}^+$ we obtain:

$$
y(t) = g(t) + e^{i\omega t}(i\omega - a) \int_0^t g(\tau)e^{-i\omega\tau}\,d\tau. \qquad (30)
$$

If we let now $t \to +\infty$ and take into account that (26) converges pointwise almost everywhere while using the relation $(i\omega - a)\mathcal{L}(g)(i\omega) = H(i\omega)$, we find that for almost every ω

$$
y(t) = g(t) + H(i\omega)e^{i\omega t} + o(1), \qquad t \to +\infty. \qquad (31)
$$

which is the formula we are aiming for. It says that, for almost every ω, the output $y(t)$ asymptotically winds at constant angular speed ω on the boundary of a tubular neighborhood of $t \mapsto g(t)$ having radius $|H(i\omega)|$. In general, the behavior at exceptional frequencies for which (31) does not hold can be quite chaotic. Note that g depends on a but not on ω, hence (31) shows that the limiting behavior of $g(t)$ for large t depends only on the system. Whereas g needs not tend to 0 at infinity[4] it is nevertheless small "most of the time" since obviously

$$m\Big(\{t:\ t\geq t_0,\ |g(t)|>\alpha\}\Big)\leq \frac{\|g\|^2_{L^2([t_0,+\infty))}}{\alpha^2}$$

which goes to 0 as $t_0\to+\infty$. Thus, asymptotically in time, the output resulting from a periodic input with pulsation ω is "most of the time and for almost every ω" close to a signal having same period, gain $|H(i\omega)|$, and phase shift $\arg H(i\omega)$. When g tends to zero at infinity, the result is neater as the words "most of the time" can be omitted from the previous sentence. This occurs for instance if $\Phi\in L^1(\mathbb{R}^+)$, as follows easily by dominated convergence from the formula $g=\Phi*(\chi_{\mathbb{R}^+}(.)e^{a\cdot})$; in this case (31) holds in fact for every ω.

Difficulties connected with the asymptotic behavior of g disappear if instead of $\chi_{\mathbb{R}^+}(t)e^{i\omega t}$ we use an input achieving a smooth transition between the zero function and $t\mapsto e^{i\omega t}$, like $u(t)=\phi_1(t)e^{i\omega t}$ where $\phi_1(t)=\int_0^t\varphi(\tau)d\tau$ was defined earlier. Then, taking into account that $\operatorname{supp}\varphi\subset[-1,1]$, a computation similar to (29) with $\Phi_{\varepsilon,T}(t)$ replaced by $\phi(t)\phi_\varepsilon(T-t)$ leads to

$$y(t)=(g*\varphi)(t)++e^{i\omega t}(i\omega-a)\left(\int_0^{t-1}g(\tau)e^{-i\omega\tau}\,d\tau+\int_{t-1}^{t+1}g(\tau)e^{-i\omega\tau}\varphi(t-\tau)\,d\tau\right)\qquad(32)$$

instead of (30). Because $g\in L^2(\mathbb{R}^+)$, both $(g*\varphi)(t)$ and the last integral in (32) tend to zero as $t\to+\infty$ by the Schwarz inequality, so the term $g(t)$ is no longer present in (31). It is worth observing that a similar conclusion is reached when $\Phi\in L^2(\mathbb{R}^+)$, a condition that does not subsume stability but allows for some unstable systems.

This interpretation of the transfer function as an asymptotic multiplier frequency-wise in response to harmonic inputs is of great importance in design.

References

[1] Laurent Baratchart. *Sur l'approximation rationnelle L^2 pour les systèmes dynamiques linéaires.* thèse de doctorat ès sciences, Université de Nice, Septembre 1987.

[2] Belevitch. *Classical network theory.* series in information systems. Holden Days, 1968.

[3] Herbert Carlin and Pier Paolo Civalleri. *Wideband circuit design.* Electronic engineering systems series. CRC Press, 1998.

[4] H.J. Carlin and D.C. Youla. Network synthesis with negative resistors. *Proceedings of the IRE*, 49(5):907–920, 1961.

[5] Pierre Faure, Michel Clerget, and cois Germain, Fran˙ *Opérateurs rationnels positifs.* Méthodes Mathématiques de l'Informatique. Dunod, 1979.

[6] Loukas Grafakos. *Modern Fourier analysis.* Number 250 in Grad. Texts in Maths. Springer, 2009.

[7] Kenneth Hoffman. *Banach spaces of analytic functions.* series in modern analysis. Prentice-Hall, 1962.

[4]If we let $\Phi=\sum_{k\in\mathbb{N}}\chi_{\mathbb{R}^+}(.-t_k)e^{\alpha_k(.-t_k)}$ with $\alpha_k=b+i\beta_k$ where $b<0$ and $\beta_k\in\mathbb{R}$, then $H(s)=\sum_k e^{-t_ks}/(s-\alpha_k)$ will lie in $\mathcal{H}^\infty$ if $|\beta_k|$ tends to infinity sufficiently fast but $g=\Phi*(\chi_{\mathbb{R}^+}(.)e^{a\cdot})$ does tend to zero at infinity. A real-valued example is obtained upon taking the α_k in conjugate pairs.

[8] K. Kurokawa. Power waves and the scattering matrix. *Microwave Theory and Techniques, IEEE Transactions on*, 13(2):194–202, Mar 1965.

[9] O. P. Misra and J.L. Lavoine. *Transform analysis of generalized functions*, volume 119 of *Mathematics Studies*. North-Holland, 1986.

[10] Jonathan Partington. *Linear operators and linear systems*. Number 60 in Student texts. London Math. Soc., 2004.

[11] Walter Rudin. *Real and complex analysis*. Mc Graw-Hill, 1982.

[12] Walter Rudin. *Functional Analysis*. International Series in Pure and Applied Mathematics. Mc Graw-Hill, 2 edition, 1991.

[13] P. Ziemer, William. *Weakly Differentiable Functions*. Number 120 in Grad. Texts in Maths. Springer, 1989.

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